Riemann Hypothesis and Physics

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Abstract

Riemann hypothesis states that the nontrivial zeros of Riemann Zeta function lie on the axis $x = 1/2$. Since Riemann zeta function allows interpretation as a thermodynamical partition function for a quantum field theoretical system consisting of bosons labelled by primes, it is interesting to look Riemann hypothesis from the perspective of physics. Quantum TGD and also TGD inspired theory of consciousness provide additional view points to the hypothesis and suggests sharpening of Riemann hypothesis, detailed strategies of proof of the sharpened hypothesis, and heuristic arguments for why the hypothesis is true.

The idea that the evolution of cognition involves the increase of the dimensions of finite-dimensional extensions of p-adic numbers associated with p-adic space-time sheets emerges naturally in TGD inspired theory of consciousness. A further input that led to a connection with Riemann Zeta was the work of Hardmuth Mueller [34] suggesting strongly that e and its $p-1$ powers at least should belong to the extensions of p-adics. The basic objects in Mueller's approach are so called logarithmic waves $exp(iklog(u))$ which should exist for $u = n$ for a suitable choice of the scaling momenta k.

Logarithmic waves appear also as the basic building blocks (the terms $n^s = exp(log(n)(Re[s]+iIm[s]))$ in Riemann Zeta. This inspires naturally the hypothesis that also Riemann Zeta function is universal in the sense that it is defined at is zeros $s = 1/2+iy$ not only for complex numbers but also for all p-adic number fields provided that an appropriate finite-dimensional extensions involving also transcendentals are allowed. This allows in turn to algebraically continue Zeta to any number field. The zeros of Riemann zeta are determined by number theoretical quantization and are thus universal and should appear in the physics of critical systems. The hypothesis $log(p) = \frac{q_1(p)exp[q_2(p)]}{\pi}$ explains the length scale hierarchies based on powers of e, primes p and Golden Mean.

Mueller's logarithmic waves lead also to an elegant concretization of the Hilbert Polya conjecture and to a sharpened form of Riemann hypothesis: the phases q^{-iy} for the zeros of Riemann Zeta belong to a finite-dimensional extension of R_p for any value of primes q and p and any zero $1/2 + iy$ of ζ. The question whether the imaginary parts of the Riemann Zeta are linearly independent (as assumed in the previous work) or not is of crucial physical significance. Linear independence implies that the spectrum of the super-canonical weights is essentially an infinite-dimensional lattice. Otherwise a more complex structure results. The numerical evidence supporting the translational invariance of the correlations for the spectrum of zeros together with p-adic considerations leads to the working hypothesis that for any prime p one can express the spectrum of zeros as the product of p^{th} powers

for a subset of Pythagorean prime phases and p^{th} power U^p of a fixed subset U of roots of unity. The spectrum of zeros could be expressed as a union over the translates of the same basic spectrum defined by the roots of unity translated by the phase angles associated with p^{th} powers of a subset of Pythagorean phases: this is consistent with what the spectral correlations strongly suggest. That decompositions defined by different primes p yield the same spectrum would mean a powerful number theoretical symmetry realizing p-adicities at the level of the spectrum of Zeta.

A second strategy is based on, what I call, Universality Principle. The function, that I refer to as $\hat{\zeta}$, is defined by the product formula for ζ and exists in the infinite-dimensional algebraic extension Q_∞ of rationals containing all roots of primes. $\hat{\zeta}$ is defined for all values of s for which the partition functions $1/(1-p^{-z})$ appearing in the product formula have value in Q_∞. Universality Principle states that $|\hat{\zeta}|^2$, defined as the product of the p-adic norms of $|\hat{\zeta}|^2$ by reversing the order of producting in the adelic formula, equals to $|\zeta|^2$ and, being an infinite dimensional vector in Q_∞, vanishes only if it contains a rational factor which vanishes. This factor is present only provided an infinite number of partition functions appearing in the product formula of $\hat{\zeta}$ have rational valued norm squared: this locates the plausible candidates for the zeros on the lines $Re[s] = n/2$.

Universality Principle implies the following stronger variant about sharpened form of the Riemann hypothesis: the real part of the phase p^{-iy} is rational for an infinite number of primes for zeros of ζ. Universality Principle, even if proven, does not however yield a proof of the Riemann hypothesis. The failure of the Riemann hypothesis becomes however extremely implausible. An important outcome of this approach is the realization that super-conformal invariance is a natural symmetry associated with ζ (not surprisingly, since the symmetry group of complex analysis is in question!).

Super-conformal invariance inspires a strategy for proving the Riemann hypothesis. The vanishing of the Riemann Zeta reduces to an orthogonality condition for the eigenfunctions of a non-Hermitian operator D^+ having the zeros of Riemann Zeta as its eigenvalues. The construction of D^+ is inspired by the conviction that Riemann Zeta is associated with a physical system allowing super-conformal transformations as its symmetries and second quantization in terms of the representations of the super-conformal algebra. The eigenfunctions of D^+ are analogous to coherent states of a harmonic oscillator and in general they are not orthogonal to each other. The states orthogonal to a vacuum state (having a negative norm squared) correspond to the zeros of Riemann Zeta. The physical states having a positive norm squared correspond to the zeros of Riemann Zeta at the critical line. Riemann hypothesis follows both from the hermiticity and posi-

tive definiteness of the metric in the space of states corresponding to the zeros of ζ. Also conformal symmetry in appropriate sense implies Riemann hypothesis and after one year from the discovery of the basic idea it became clear that one can actually construct a rigorous twenty line long analytic proof for the Riemann hypothesis using a standard argument from Lie group theory.

1 Introduction

Riemann hypothesis states that the nontrivial zeros of Riemann Zeta function lie on the axis $x = 1/2$. Since Riemann zeta function allows interpretation as a thermodynamical partition function for a quantum field theoretical system consisting of bosons labelled by primes, it is interesting to look Riemann hypothesis from the perspective of physics. Quantum TGD and also TGD inspired theory of consciousness provide additional view points to the hypothesis and suggests sharpening of Riemann hypothesis, detailed strategies of proof of the sharpened hypothesis, and heuristic arguments for why the hypothesis is true.

The idea that the evolution of cognition involves the increase of the dimensions of finite-dimensional extensions of p-adic numbers associated with p-adic space-time sheets emerges naturally in TGD inspired theory of consciousness. A further input that led to a connection with Riemann Zeta was the work of Hardmuth Mueller [34] suggesting strongly that e and its $p-1$ powers at least should belong to the extensions of p-adics. The basic objects in Mueller's approach are so called logarithmic waves $exp(iklog(u))$ which should exist for $u = n$ for a suitable choice of the scaling momenta k.

Logarithmic waves appear also as the basic building blocks (the terms $n^s = exp(log(n)(Re[s] + iIm[s]))$ in Riemann Zeta. This inspires naturally the hypothesis that also Riemann Zeta function is universal in the sense that it is defined at is zeros $s = 1/2 + iy$ not only for complex numbers but also for all p-adic number fields provided that an appropriate finite-dimensional extensions involving also transcendentals are allowed. This allows in turn to algebraically continue Zeta to any number field. The zeros of Riemann zeta are determined by number theoretical quantization and are thus universal and should appear in the physics of critical systems. The hypothesis $log(p) = \frac{q_1(p)exp[q_2(p)]}{\pi}$ explains the length scale hierarchies based on powers of e, primes p and Golden Mean.

Mueller's logarithmic waves lead also to an elegant concretization of the Hilbert Polya conjecture and to a sharpened form of Riemann hypothesis: the phases q^{-iy} for the zeros of Riemann Zeta belong to a finite-dimensional extension of R_p for any value of primes q and p and any zero $1/2 + iy$ of ζ. The question whether the imaginary parts of the Riemann Zeta are linearly independent (as assumed in the previous work) or not is of crucial physical significance. Linear independence implies that the spectrum of the super-canonical weights is essentially an infinite-dimensional lattice. Otherwise a more complex structure results. The numerical evidence supporting the translational invariance of the correlations for the spectrum of zeros together with p-adic considerations leads to the working hypothesis that for any prime p one can express the spectrum of zeros as the product of a subset of Pythagorean phases and of a fixed subset U of roots of unity. The spectrum of zeros could be expressed as a union over the translates of the same basic spectrum defined by the roots of unity translated by the phase angles associated with a subset of Pythagorean phases: this is consistent with what the spectral correlations strongly suggest. That decompositions defined by different primes p yield the same spectrum would mean a powerful number theoretical symmetry realizing p-adicities at the level of the spectrum of Zeta.

A second strategy is based on, what I call, Universality Principle. The function, that I refer to as $\hat{\zeta}$, is defined by the product formula for ζ and exists in the infinite-dimensional algebraic extension Q_∞ of rationals containing all roots of primes. $\hat{\zeta}$ is defined for all values of s for which the partition functions $1/(1-p^{-z})$ appearing in the product formula have value in Q_∞. Universality Principle states that $|\hat{\zeta}|^2$, defined as the product of the p-adic norms of $|\hat{\zeta}|^2$ by reversing the order of producting in the adelic formula, equals to $|\zeta|^2$ and, being an infinite dimensional vector in Q_∞, vanishes only if it contains a rational factor which vanishes. This factor is present only provided an infinite number of partition functions appearing in the product formula of $\hat{\zeta}$ have rational valued norm squared: this locates the plausible candidates for the zeros on the lines $Re[s] = n/2$.

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Super-conformal invariance inspires a strategy for proving the Riemann hypothesis. The vanishing of the Riemann Zeta reduces to an orthogonality condition for the eigenfunctions of a non-Hermitian operator D^+ having the zeros of Riemann Zeta as its eigenvalues. The construction of D^+ is inspired by the conviction that Riemann Zeta is associated with a physical system allowing super-conformal transformations as its symmetries and second quantization in terms of the representations of the super-conformal algebra. The eigenfunctions of D^+ are analogous to coherent states of a harmonic oscillator and in general they are not orthogonal to each other. The states orthogonal to a vacuum state (having a negative norm squared) correspond to the zeros of Riemann Zeta. The physical states having a positive norm squared correspond to the zeros of Riemann Zeta at the critical line. Riemann hypothesis follows both from the hermiticity and positive definiteness of the metric in the space of states corresponding to the zeros of ζ. Also conformal symmetry in appropriate sense implies Riemann hypothesis and after one year from the discovery of the basic idea it became clear that one can actually construct a rigorous twenty line long analytic proof for the Riemann hypothesis using a standard argument from Lie group theory.

These approaches concretize the vision about TGD based physics as a generalized number theory. Two new realizations of the super-conformal algebra result and the second realization has direct application to the modelling of $1/f$ noise. The zeros of ζ code for the states of an arithmetic quantum field theory coded also by infinite primes: also the hierarchical structure of the many-sheeted space-time is coded. Even some basic quantum numbers of particles of TGD Universe might have number theoretical representation.

2 General vision

Quantum TGD has inspired several strategies of proof of the Riemann hypothesis. The first strategy is based on the modification of Hilbert Polya hypothesis by requiring that the physical system in question has super-conformal transformations as its symmetries. Second strategy is based on considerations based on TGD inspired quantum theory of cognition and a generalization of the number concept inspired by it. Together with some physical inputs one ends up to a hypothesis that Riemann Zeta is well defined in all number fields near its zeros provided finite-dimensional extensions of p-adic numbers are allowed. This hypothesis generalizes the earlier hypothesis assuming that the extensions are trivial or at most algebraic. Third strategy is based on, what I call, Universality Principle.

There are also strong physical motivations to say something explicit about the spectrum of zeros and here p-adicization program inspires the hypothesis the numbers q^{iy}, q prime, belong to a finite algebraic extension of p-adic number field R_p for every prime p. The findings about the correlations of the spectrum of zeros inspire very concrete hypothesis about the spectrum of zeros as a union of translates of the same basic spectrum and this hypothesis is supported by the physical identification of the zeros of Zeta as super-canonical conformal weights.

2.1 Generalization of the number concept and Riemann hypothesis

The hypothesis about p-adic physics as physics of cognition leads to a generalization of the notion of number obtained by gluing reals and various p-adic number fields together along rational numbers common to all of them. This structure is visualizable as a book like structure with pages represented by the number fields and the rim of the book represented by rationals. Even this structure can be generalized by allowing all finite-dimensional extensions of p-adic numbers including also those containing transcendental numbers and performing similar identification. Kind of fractal book might serve as a visualization of this structure.

In TGD inspired theory of consciousness intentions are assumed to correspond to quantum jumps involving the transformation of p-adic space-time sheets to real ones. An intuitive expectation is p-adic and real space-time sheets to each other must have a maximum number of common rational points. The building of idealized model for this transformation leads to the problem of defining functions having Taylor series with rational coefficients and continuable to both real and p-adic functions from a subset of rational numbers (or points of space-time sheet with rational coordinates). In this manner one ends up with the hypothesis that p-adic space-time sheets correspond to finite-dimensional extensions of p-adic numbers, which can involve also transcendental numbers such as e. This leads to a series of number theoretic conjectures.

The idea that the evolution of cognition involves the increase of the dimensions of finite-dimensional extensions of p-adic numbers associated with p-adic space-time sheets emerges naturally in TGD inspired theory of consciousness. A further input that led to a connection with Riemann Zeta was the work of Hardmuth Mueller [34] suggesting strongly that e and its $p-1$ powers at least should belong to extensions of p-adics. The basic objects in Mueller's approach are so called logarithmic waves $exp(iklog(u))$ which should exist for $u = n$ for a suitable choice of the scaling momenta k.

Logarithmic waves appear also as the basic building blocks (the terms

$n^s = exp(log(n)(Re[s] + iIm[s]))$ in Riemann Zeta. This inspires naturally the hypothesis that also Riemann Zeta function is universal in the sense that it is defined at is zeros $s = 1/2 + iy$ not only for complex numbers but also for all p-adic number fields provided that an appropriate finite-dimensional extensions involving also transcendentals are allowed. This allows in turn to algebraically continue Zeta to any number field. The zeros of Riemann zeta are determined by number theoretical quantization and are thus universal and should appear in the physics of critical systems. A hierarchy of number theoretical conjectures stating that a finite number of iterated logarithms about transcendentals appearing in the extension forms a closed system under the operation of taking logarithms. Mueller's logarithmic waves lead also to an elegant concretization of the Hilbert Polya conjecture and to a sharpened form of Riemann hypothesis: the complex numbers p^{-iy} for the zeros of Riemann Zeta belong to a finite-dimensional extension of R_p for any value of p and any zero $1/2 + iy$ of ζ.

2.2 Modified form of Hilbert-Polya hypothesis

Super-conformal invariance inspires a strategy for proving (not a proof of, as was the first over-optimistic belief) the Riemann hypothesis. The vanishing of Riemann Zeta reduces to an orthogonality condition for the eigenfunctions of a non-Hermitian operator D^+ having the zeros of Riemann Zeta as its eigenvalues. The construction of D^+ is inspired by the conviction that Riemann Zeta is associated with a physical system allowing super-conformal transformations as its symmetries and second quantization in terms of the representations of super-conformal algebra. The eigenfunctions of D^+ are analogous to the so called coherent states and in general not orthogonal to each other. The states orthogonal to a vacuum state (having a negative norm squared) correspond to the zeros of Riemann Zeta. The physical states having a positive norm squared correspond to the zeros of Riemann Zeta at the critical line and possibly those having $Re[s] > 1/2$.

A possible proof of the Riemann hypothesis by reductio ad absurdum results if one assumes that the states corresponding to zeros of ζ span a space with a hermitian metric. Riemann hypothesis follows both from the hermiticity and positive definiteness of the metric in the space of states corresponding to the zeros of ζ. Also conformal invariance in appropriate sense implies Riemann hypothesis. Indeed, a rather rigorous proof of Riemann hypothesis results from the observation that certain generator of conformal algebra permutes the two zeros located symmetrically with respect to the critical line. If the action of this generator exponentiates, Riemann hypothesis follows since exponentiation would imply the existence of infinite number of zeros along a line parallel to $Re[s]$-axis. One can formulate this argument rigorously using first order differential equation, and if one forgets all the preceiding refined philosophical arguments, one can prove Riemann hypothesis using twenty line long analytic argument! Perhaps Ramajunan could have made this!

As already noticed, the state space metric can be made positive definite provided Riemann hypothesis holds true. Thus the system in question might quite well serve as a concrete physical model for quantum critical systems possessing super-conformal invariance as both dynamical and gauge symmetry.

2.3 Universality Principle

The function, what I call $\hat{\zeta}$, is defined by the product formula for ζ and exists in the infinite-dimensional algebraic extension of rationals containing all roots of primes. $\hat{\zeta}$ is defined for all values of s for which the partition functions $1/(1-p^{-s})$ appearing in the product formula have value in the algebraic extension. Universality Principle states that $|\hat{\zeta}|^2$, defined as the product of the p-adic norms of $|\hat{\zeta}|^2$ by reversing the order of producting in the adelic formula, equals to $|\zeta|^2$ and, being an infinite dimensional vector in the algebraic extension of the rationals, vanishes only if it contains a rational factor which vanishes. This factor is present only provided an infinite number of partition functions appearing in the product formula of $\hat{\zeta}$ have rational valued norm squared: this locates the plausible candidates for the zeros on the lines $Re[s] = n/2$.

Universality Principle generalizes the original sharpened form of the Riemann hypothesis: the real parts of the phases p^{-iy} are rational. Universality Principle, even if proven, does not however yield a proof of the Riemann hypothesis. The failure of Riemann hypothesis becomes however extremely implausible and one could consider the possibility of regarding Riemann Hypothesis as an axiom. An important outcome of this approach is the realization that super-conformal invariance is a natural symmetry associated with Riemann Zeta (not surprisingly, since the symmetry group of complex analysis is in question!).

2.4 Physics, Zetas, and Riemann Zeta

Although the original naive speculations are probably not correct, the work with Riemann Zeta led to several new mathematical concepts and rather concrete ideas about how physics in TGD Universe might reduce to generalized number theory.

2.4.1 Do M- and U-matrices exist in all number fields simultaneously?

TGD predicts two kinds of fundamental matrices [C1, C2]. S-matrix of particle physics is replaced with M-matrix defining time-like entanglement coefficients between positive and negative energy parts of zero energy states (all conserved quantum numbers vanish for these states so that they are creatable from vacuum). M-matrix equals to the product of a square root of density matrix and unitary matrix and cannot have elements between different number fields. U-matrix characterizes the unitary process associated with quantum jump between zero energy states. Therefore U can have elements also between different number fields and should be number theoretically universal. U-matrix would describe quantum jumps describing a transformation of intention to action for instance, or transformation of zero energy state to pure cognition.

One must consider the possibility that M-matrix can be constructed independently in all number fields. On the other hand, the assumption M-matrix is continuable from a matrix whose elements are algebraic numbers is however very attractive (ordinary S-matrix has 3-momenta of particles as continuous indices). One must of course be cautious in order to avoid the situation in which the theory effectively reduces to that in the field of algebraic numbers. To achieve this pit-hole one must understand how real and p-adic physics differ from each other. p-Adic variants of light-like 3-surfaces can obey same algebraic equations as their real counterparts. Real 4-D space-time sheets serving as classical correlates of classical degrees of freedom in quantum measurement theory however obey genuine field equations and it is not at all whether their solutions allow an algebraic continuation to the p-adic context. Since it is not possible to measure cognition, one might argue that p-adic space-time sheets are not needed at all.

Both U- and S-matrices could exist in a well-defined sense simultaneously in all number fields provided finite-dimensional extensions of p-adic numbers are allowed. It is also natural to expect that the structure of the these matrices reflects the evolution of cognition as a gradual increase of the p-adic prime characterizing the space-time sheet and of the dimension of the algebraic extension involved. These matrices should have a hierarchical decomposition into increasingly complex S- and U-matrices using direct sum and direct product. One might even hope of identifying universal elementary S and U-matrices serving as basic building blocks in this construction so that a number-theoretical bootstrap might make sense.

2.4.2 Do conformal weights of the generators of super-canonical algebra correspond to zeros of some zeta function?

For long time the zeros of Riemann Zeta remained excellent candidates for the conformal weights labelling the generators of super-canonical algebra [B2, B3, B4]. The basic motivation was that the radial conformal weights have very naturally real part which equals to -1/2 as does also the negative of the real part of complex zeros of Riemann Zeta. Also other conformal weights are possible but not so natural.

1. Why Riemann Zeta does not work

The following observations have however changed the situation.

a) The almost defining property of zeta functions is that their complex zeros reside at the critical line. There exists a lot of zeta functions [E3] so that the spectrum of super-canonical conformal weights allows to consider also other zetas.

b) The zeta functions analogous to the basic building blocks of Riemann Zeta labelled by prime p are especially natural from the point of view of p-adic length scale hypothesis and they have automatically the nice algebraic properties required by the number theoretic universality whereas in the case of Riemann Zeta they must be conjectured.

c) The generalized eigenvalues of the modified Dirac operator define in a very natural manner zeta functions coding geometric information about partonic 2-surfaces whereas Riemann Zeta has no obvious interpretation of this kind.

These findings do not of course exclude Riemann zeta or zetas analogous to it. For instance, one can assign Riemann Zeta to the purely bosonic infinite primes very naturally. The spectrum of the scaling generator L_0 consists of non-negative integers and the positive part of spectrum defines a zeta function of form $\sum_{n>0} g(n)n^{-s}$, which might be relevant for quantum TGD. I do not known about the zeros of this zeta function.

A further natural speculation was that the zeros of polyzetas $\zeta(z_1, ..., z_K)$ label the super-canonical conformal weights of K-particle bound states. The vanishing of loop corrections could be understood as being due to the fact that they are proportional to polyzetas having super-canonical conformal weights as arguments. This speculation was inspired by the fact that polyzetas with integer arguments emerge in loop corrections of quantum field theories.

2. Zeta functions assignable to the modified Dirac operator

In the case of the modified Dirac operator and super-canonical conformal weights Riemann Zeta is naturally replaced by a zeta function determined by purely physical considerations (detailed argument can be found in [B4, C1]).

a) The determinant of the modified Dirac operator D gives rise to the vacuum functional of TGD and the conjecture is that it reduces to a product of exponents of Kähler function and Chern-Simons action. The construction assigns to a given 3-D light-like surface X^3_l a 4-D space-time sheet conjectured to be a preferred extremal of Kähler action [B4].

b) The generalized eigenvalue λ of D is actually a scalar field depending on the coordinates of partonic 2-surface X^2 (and light-like 3-surface X^3_l). λ codes purely geometric information about the light-like 3-surface, and Higgs vacuum expectation is naturally proportional to λ.

c) The minima of the modulus of the holomorphic function λ in X^2 give rise to what I call number theoretic braids. Dirac determinant is product of the eigenvalues at the minima of $|\lambda|$ interpreted as a function X^3_l.

d) One can assign to the values of λ at the points of the number theoretic braid also zeta function, call it ζ. ζ codes geometric information about 3-surface and super-canonical conformal weights correspond naturally to its zeros. ζ is sum over a finite number of terms only, and if it is rational function of a suitable coordinate, it has all the required number theoretic properties whereas in the case of Riemann Zeta these properties require strong number theoretic conjectures.

The notion of polyzeta might generalize in a natural manner to a dynamical polyzeta. Suppose that one has a collection X^2_i of partonic 2-surfaces assignable to a connected space-like 3-surface defined by the intersection $X^3 = X^4 \cap \delta M^4_+ \times CP_2$. In this kind of situation one might hope that the notion of polyzeta generalizes and can be defined in terms of the generalized eigenvalues of the modified Dirac operator assigned with various partonic 2-surfaces X^2_i. If X^3 is connected, the polyzeta cannot be a mere product of independent zetas associated with X^2_i obtained by assigning separate space-time sheets to the light-like orbits of X^2_i. Even if it reduces to a product, the eigenvalues assignable to X^2_i are correlated by the constraint that the minimization of λ_i is consistent with the condition $X^2_i \subset X^3$. This polyzeta would naturally characterize the bound state character of the resulting state.

3. Almost dead speculation related to Riemann Zeta

The construction of scalar propagator discussed in the appendix of [C7]

(I do not have heart to throw it away yet) was based on the assumption that scalar propagator can be regarded as a partition function in super-canonical algebra. In the approximation that the imaginary parts of the zeros of linearly independent, the masses for the predicted universal spectrum of resonances is expressible in terms of zeros of Riemann Zeta. A similar universal spectrum of resonances (which are not poles but delta functions) is predicted also when $1/L_0$ replaces scalar propagator. Ultraviolet cutoff appears automatically and in p-adic context one must identify it as p-adic length scale by number theoretical existence requirement. p-Adic length hierarchy scale emerges thus naturally.

Both the known spectral correlations of Riemann Zeta and physical arguments suggest strongly that the zeros of Riemann Zeta are not actually linearly independent but that the spectrum factorizes in such a manner that for a given prime p it corresponds to product subset of p-ples of prime Pythagorean phases and p-ples of algebraic phases, the latter bringing in linear dependence. Thus the spectrum is union of translates of the basic spectrum defined by the rational phase angles by subset of phase angles associated with multiples of Pythagorean prime phases, which implies approximate translational invariance of the spectral correlation functions.

A further prediction was that a hierarchy of propagators results and is labelled by the the hierarchy of sets $\{y_1 < y_2 < ... < y_K\}$ of imaginary parts $y_i > 0$ of the non-trivial zeros of Zeta ordered by there magnitude. This cutoff hierarchy corresponds to a finite p-adic phase resolution (the better the phase resolution the higher the algebraic dimension of the extension of p-adic numbers needed). The limit when all zeros of Zeta are included can exist only provided the zeros are not linearly independent. The hypothesis that q^{iy}, q any prime, belongs to a finite extension of R_p for all primes p is necessary for the p-adicization of the propagator.

These results establish the role of Riemann Zeta in the physics of TGD Universe.

2.5 General number theoretical ideas inspired by the number theoretic vision about cognition and intentionality

The following two ideas serve as guide lines in the attempt to relate cognition, intentionality and number theory to each other so that number theory would allow to construct a more detailed view about the realization of intentionality and cognition. As a matter fact, the general ideas about intention and cognition in turn generate very general number theoretical conjectures.

a) Real and p-adic number fields form a book like structure with pages represented by number fields glued together along rationals forming the rim of the book. For the extensions of p-adic numbers further common points result and the book becomes fractal if all possible extensions are allowed. This picture generalizes to the level of the imbedding space and allows to see space-time surfaces as consisting of real and p-adic space-time sheets belonging to various extensions of these numbers. This generalized view about numbers gives hopes about an unambigious definition of what some number, say e, appearing in an extension of p-adic numbers really means.

b) The first new idea is roughly that the discovery of notion of any algebraic or transcendental number x (such as Φ or e) involves a quantum jump in which there is generated a p-adic space-time sheet for which the existing finite-dimensional extension of p-adic numbers is replaced by a finite-dimensional extension involving also x. Also some higher powers of the number are involved. For instance, for e $p-1$ powers are necessarily needed (e^p exists p-adically).

c) The p-adic-to-real transition serving as a correlate for the transformation of intention to action is most probable if the number of common rational valued points for the p-adic and real space-time sheet is high. The requirement of real and p-adic continuity and even smoothness however forces upper and lower p-adic length scale cutoffs so that common points are in certain length scale range.

d) The points of M^4_+ with integer valued Minkowski coordinates using CP_2 length related fundamental length scale as a basic unit is a good guess for the subset of M^4_+ defining the rational points of the M^4_+ involved. CP_2 coordinates as functions of M^4_+ coordinates should be rational or belong to some finite-dimensional extension of p-adics. Of course, also rational points of M^4_+ are possible, and the evolution of cognition should correspond to the increase of the algebraic dimension of the extension.

e) A very powerful hypothesis is that the p-adic and real functions have the same analytic form besides coinciding at the chosen rational points defining the p-adic pseudo constant involved. Since the pseudo constant defines the corresponding real function in rational points, there are indeed good hopes that the transformation of p-adic intention to real action is possible. This assumption favors functions which allow at some point (most naturally origin) a Taylor series with rational valued Taylor coefficients.

2.5.1 Is e an exceptional transcendental?

Neper number is obviously the simplest one and only the powers e^k, $k = 1, ..., p-1$ of e are needed to define p-adic counterpart of e^x for $x = n$. In case of trigonometric functions deriving from e^{ix}, also e^i and its $p-1$ powers must belong to the extension.

An interesting question is whether e is a number theoretically exceptional transcendental or whether it could be easy to find also other transcendentals defining finite-dimensional extensions of p-adic numbers.

a) Consider functions $f(x)$, which are analytic functions with rational Taylor coefficients, when expanded around origin for $x > 0$. The values of $f(n)$, $n = 1, ..., p-1$ should belong to an extension, which should be finite-dimensional.

b) The expansion of these functions to Taylor series generalizes to the p-adic context if also the higher derivatives of f at $x = n$ belong to the extension. This is achieved if the higher derivatives are expressible in terms of the lower derivatives using rational coefficients and rational functions or functions, which are defined at integer points (such as exponential and logarithm) by construction. A differential equation of some finite order involving only rational functions with rational coefficients must therefore be satisfied (e^x satisfying the differential equation $df/dx = f$ is the optimal case in this sense). The higher derivatives could also reduce to rational functions at some step ($log(x)$ satisfying the differential equation $df/dx = 1/x$).

c) The differential equation allows to develop $f(x)$ in power series, say in origin

$$f(x) = \sum f_n \frac{x^n}{n!}$$

such that f_{n+m} is expressible as a rational function of the m lower derivatives and is therefore a rational number.

The series converges when the p-adic norm of x satisfies $|x|_p \leq p^k$ for some k. For definiteness one can assume $k = 1$. For $x = 1, ..., p-1$ the series does not converge in this case, and one can introduce and extension containing the values $f(k)$ and hope that a finite-dimensional extension results.

Finite-dimensionality requires that the values are related to each other algebraically although they need not be algebraic numbers. This means symmetry. In the case of exponent function this relationship is exceptionally simple. The algebraic relationship reflects the fact that exponential map represents translation and exponent function is an eigen function of a translation operator. The necessary presence of symmetry might mean that the situation reduces always to either exponential action. Also the phase factors $exp(iq\pi)$ could be interpreted in terms of exponential symmetry. Hence the reason for the exceptional role of exponent function reduces to group theory.

Also other extensions than those defined by roots of e are possible. Any polynomial has n roots and for transcendental coefficients the roots define a finite-dimensional extension of rationals. It would seem that one could allow the coefficients of the polynomial to be functions in an extension of rationals by powers of a root of e and algebraic numbers so that one would obtain infinite hierarchy of transcendental extensions.

2.5.2 Some no-go theorems

Elementary functions like exp(x), log(1+x), cos(x), sin(x), are obviously favored by the previous considerations, in particular by the requirement of the form invariance of the function in p-adic-to-real transition. They indeed have p-adic Taylor expansion which converges for $|x|_p < 1$. The definition at integer valued points for which $x \ mod \ p = n$, $n = 0, 1, ..., p - 1$, requires the introduction of an extension of p-adic numbers. The natural first guess is that this extension is finite-dimensional. Of course, this is just a hypothesis to be discussed and motivated by the idea that p-adic extensions reflect our own finite intelligence.

1. *Can powers of $log(p)$ define a finite-dimensional extension of p-adics?*

The number theoretical entropy associated with any p-adic prime for which the ordinary logarithm $log(p_n)$ is replaced by the logarithm of the p-adic norm of p_n, is proportional to a $log(p)$-factor. As already noticed, if bit is used as unit, then only the rationality of $log(p)/log(2)$ would be needed and $log(p)$ need not correspond to a finite-dimensional extension of p-adics. Unfortunately, also this conjecture turns out to be false.

The first observation is that $log(1 + x)$, $x = O(p)$ exists as an ordinary p-adic number and the logarithm of $log(m)$, $m < p$ such that the powers of m span the numbers $1, ..., p - 1$ besides $log(p)$ need be introduced to the extension in order that logarithm of any integer and in fact of any rational number exists p-adically. The problem is however that the powers of $log(m)$ and $log(p)$ might generate an infinite-dimensional extension of p-adic numbers.

First some no-go theorems inspired by wishful conjectures (professional number theorists must regard me as an idiot!).

a) $log(p) = q/t$, where t is a fixed transcendental number, say π, cannot hold true. The reason is that the rationality of $log(p_1)/log(p_2) = q_1/q_2 = r/s$ implies that $p_1^s = p_2^r$ in contradiction with the prime number property of p_1 and p_2. This excludes also the rationality of $log(q_1)/log(q_2)$. It is however possible to have *single* rational q for which say $\pi/log(q)$ is rational.

b) $log(q)$, q prime, cannot correspond to a finite dimensional extension of R_p in the sense that a finite power of $log(q)$ would be a rational number. Assume that this is the case, i.e. $(log(q))^{m_{p,q}} = x_{p,q}$, where $x_{p,q}$ is an ordinary p-adic number in R_p, and assume that e belongs to extension. For definiteness let us assume $|x_{p,q}| < 1$ and write

$$q = exp(log(q)) = \sum_n log(q)^n/n! = \sum_{k=0}^{m-1} c_k log(q)^k \ \ , \ \ c_k = \sum_n \frac{x_{p,q}^n}{(k+nm_{p,q})!} \ \ .$$

The righthand side gives m terms corresponding to the m powers of $log(q)$ and only the lowest term can be non-vanishing and equals to q. The convergence of series requires that $x_{p,q}$ has p-adic norm smaller than one. This however implies that lowest order term has p-adic norm equal to one. For $q = p$ this leads to contradiction since one would have $p = 1 + O(p)$. For $|x_{p,q}|_p \geq 1$ the argument fails since the expansion does not make sense. For $q = exp(p^k log(q))$, k sufficiently large, the expansion exists and in this case one as $q^{p^k} = 1 + O(p)$, which for $q = p$ gives a contradiction.

c) One might hope that $log(p)$ belongs to an extension containing e or its root, or in the most general case root of a polynomial with coefficients which belongs to an extension of rationals by e and algebraic numbers. For instance, the ansatz $log(p) = e^{q_1(p)}q_2(p)$ with $q_2(p_1) \neq q_2(p_2)$ for all pairs of primes, would guarantee that logarithms belong to a finite-dimensional extension. There are no problems with the prime property as is clear from the expression

$$p_1 = p_2^{[exp(q_1(p_1)-q_1(p_2)]\times \frac{q_2(p_1)}{q_2(p_2)}} \ \ .$$

From the assumption it follows that the exponent cannot reduce to a rational number.

Unfortunately the ansatz does not work! One can write

$$p_1 = exp\left(e^{q_1(p_1)}q_2(p_1)\right)$$

and for those primes p_2 whose positive power divides $q_2(p_1)$, one can expand the exponential in a converging power series in powers of a root of e, and one obtains that ordinary p-adic number is expressible as a non-trivial combination of powers of a root of e.

e) Obviously one must give up hopes for obtaining a finite-dimensional extension for the logarithms. Also the hope that $log(p)/log(2)$ is always rational guaranteing that p-adic entropy would be always rational multiple of bit must be given up. There could however exist single rational for which $log(q)/pi$ is rational. In fact, the rather speculative considerations related to Kähler couplings strength inspire the question whether the number $log[(2^{127}-1)\times 2\times 3\times 5\times 7\times 11\times 13\times 17\times 19\times 23]/\pi$ could be rational [C6]. If this conjecture were true it would fix completely the p-adic evolution of Kähler coupling strength.

3. π cannot belong to a finite-dimensional extension of p-adic numbers

A simple argument excludes the possibility that π could belong to some finite-dimensional extension $\pi = \sum c_n e_n$. If this is the case one can write $exp(ip^k\pi) = -1$ as a converging Taylor expansion in powers of p for high enough value of k, and the coefficients of all e_n expect $e_0 = 1$ must vanish. Since the terms in this series come in powers of p it is highly implausible that they could sum up to zero. In fact, even the coefficient of $e_0 = 1$ has wrong sign. By considering more general numbers $exp(iq\pi)$ one obtains that the expansion in terms of e_i equals to the expression of phase in infinite number of different algebraic extensions. Thus it seems obvious that π cannot belong to a finite extension.

2.5.3 Does the integration of complex rational functions lead to rationals extended by a root of e and powers of π?

These cold showers suggest that the best one might hope is that the numbers like $log(p)$ and $log(\Phi)$ could be proportional to some power π with a coefficient which belongs to a finite extension of p-adic numbers containing e. This might make it possible to continue the theory to p-adic context and also make very strong predictions.

The elementary differential and integral calculus provides important hints for as how to proceed. Derivation takes rational functions to rational functions unlike integration since the integrals of $1/x$ and and $1/(1+x^2)$ give $log(x)$ and $arctan(x)$ leading outside the realm of rational numbers. One can go to complex plane and consider the integrals of complex rational functions with complex rational coefficients and here one encounters integrals over closed curves and between two points. The rational approach is to consider rational complex plane, and first restrict to Gaussian integers which allow primes.

i) The first observation is that residy calculus for rational functions gives always integrals which are of form $2\pi iq$, q a rational number.

ii) The integral $I = \int_a^b dz/z$, $a = m_1 + in_1$, $b = m_2 + in_2$ in turn gives

$$I = log(a/b) = \frac{1}{2}\left(log(m_2^2+n_2^2) - log(m_1^2+n_1^2)\right)$$

$$+i(arctan(n_2/m_2) - arctan(n_1/m_1)) \ .$$

a) The strongest hypothesis would be that logarithm and arctan are also rationally proportional to π so that all integrals of this kind lead to an infinite-dimensional transcendental extension of p-adic numbers containing π. The strong hypothesis cannot be correct. Consider arcus tangent as an example. $arctan(m/n) = r\pi/s$ would imply $tan(r\pi/s) = m/n$, and this cannot hold true since it would imply that s:th powers of Gaussian integer $n+im$ would give an ordinary integer. This would be also true for Gaussian primes and the decomposition of Gaussian integers as products of Gaussian primes would become non-unique. There is this kind of uniqueness but this is due the units $exp(i\pi/4)$ and its powers. Indeed, $arctan(1) = \pi/4$ and proportional to π.

b) One can overcome this difficulty by replacing the ansatz with

$$arctan(q) = e^{q_1(q)}q_2\pi$$

such that $q_1(q)$ is non-vanishing for $q \neq \pm 1 \pm i$ corresponding to the units of Gaussian primes. This ansatz is completely analogous to the ansatz for $log(p)$. The beauty of this ansatz would be that the imaginary parts for the integral of $1/(z-z_0)$ between complex rational points would be proportional to π irrespective of whether the integration is over a closed or open curve. The real parts of complex integrals in turn would be proportional to $1/\pi$ of $log(p) \propto 1/\pi$ ansatz holds true.

The requirement that complex integrals are powers of π could also mean quantization of topology in TGD framework. For instance, the conformal equivalence classes of Riemann surfaces of genus g are represented by period integrals of 1-forms defining elements of cohomology group H^1 over the circles representing the elements of homology group H_1. Restricting the cohomology to a rational cohomology, the periods with standard normalization would be quantized to complex rationals multiplied by a power of π. For surfaces characterized by a given power of π one might perhaps perform the p-adicization finite-dimensionally by suitable normalizations by powers of π.

2.5.4 Why should one have $p = q_1exp(q_2)/\pi$?

There are good physical arguments suggesting that $log(p)$ should be proportional to $1/\pi$.

a) π appears naturally in the plane wave solutions of field equations $exp(in\pi u)$, $u = x/L$). These phases are well defined in a finite-dimensional algebraic extension if x/L is rational. One can however consider also logarithmic plane waves

$$exp(iku), \;\; u = log(x/L) \;\; ,$$

and ask under what conditions they are well defined and in particular, under what conditions the real/imaginary parts of these plane waves can have zeros at $u = e^n$ required by Mueller's hypothesis [34]. Mueller's hypothesis implies that $exp(ikn)$ has zeros so that $k = q\pi$ must hold true. Thus one obtains essentially ordinary plane waves.

If one has $u = q_1 e^n$, q_1 rational, one obtains also the exponential $exp(iq\pi log(q_1))$. From the point of view of p-adicization program it would be very nice if also this exponent would exist p-adically. This is guaranteed if one has

$$log(p) = \frac{q_1(p)exp\left[q_2(p)\right]}{\pi}$$

for every prime p. One can write

$$exp(iq\pi u) = exp\left[iqq_1(p)exp(q_2(p))\right] \;\; .$$

The exponential exists for those primes p_1 for which the exponent is divisible by a positive power of p_1. This means quantization conditions favoring selected primes p_1 or alternatively scaling momenta q. An easy manner to satisfy these conditions is to assume that q is a multiple of a power of p.

c) Besides Mueller's hierarchy in powers of e there are also p-adic hierarchies and the hierarchies associated with Golden Mean and one can look whether these hierarchies are obtained for suitable logarithmic waves. For $u = x/L = mp^n$ the scaling wave reads

$$exp(iku) = exp\left[iknlog(p)\right] exp\left[iklog(m)\right] \;\; .$$

For $log(p) = q_1(p)exp\left[q_2(p)\right]/\pi$ the existence of nodes for the the first factor requires $k = q\pi^2 exp\left[-q_2(p)\right]$. The second factor exists only for $m = 1$ so that nodes are possible only at $u = p^n$.

Note that $k = q\pi$ for e so that these length scale hierarchies are distinguishable number theoretically. This assumption implies that also the second exponential of product can exist in a finite-dimensional algebraic extension and can have even nodes. For the hierarchy defined by powers of Golden Mean the assumption $log(\Phi) = q_1 q exp(q_2)/\pi$ would lead to similar conclusions. Again one must leave door open for more general power of π.

2.5.5 p-Adicization of vacuum functional of TGD and infinite primes

A further input comes from TGD. The basic challenge is to continue the exponent $exp(K)$ of the Kähler function to p-adic number fields. K can be expressed as

$$K = \frac{S_K}{16\pi\alpha_K} \ ,$$

where α_K is so called Kähler coupling strength and $S_K = \int J_{\mu\nu}J^{\mu\nu}\sqrt{g}d^4x$ is Kähler action, which is essentially the Maxwell action for the induced Kähler form. The dream is that an algebraic continuation from the extensions of rational numbers defining finite extensions of p-adic numbers allows to define the theory in various number fields. The fulfillment of this dream requires that physically important quantities such as the exponent of Kähler function for CP_2 extremal and other fundamental extremals exist in a finite-dimensional extension of p-adic numbers.

1. *What is the value of Kähler coupling strength?*

The value of Kähler coupling strength is analogous to a critical temperature and can have only discrete values.

a) The discrete p-adic evolution of the Kähler coupling strength follows from the requirement that gravitational coupling constant is renormalization group invariant (see the chapter "Fusion of p-Adic and Real Variants of Quantum TGD to a More General Theory").

When combined with the requirement that the exponent of CP_2 action is a power of prime, the argument would give

$$\frac{1}{\alpha_K(p)} = \frac{4}{\pi}log(K^2) \ , \ \ K^2 = \prod_{q=2,3,...23} q \times p$$

with $\alpha_K(p = M_{127}) \simeq 136.5585$ and $\alpha/\alpha_K \simeq .9965$. Note that M_{127} corresponds to electron length scale. If the action is a rational fraction of CP_2 action, and the extension of p-adic numbers is by an appropriate root of p is enough to guarantee the existence of the Kähler function.

b) One can consider also an alternative ansatz based on the requirement that Kähler function is a rational number rather than a logarithm of a power of integer K^2. This requires an extension of p-adic numbers involving some root of e and a finite number of its powers. S_R must be rational valued using Kähler action $S_K(CP_2) = 2\pi^2$ of CP_2 type extremal as a basic unit. In fact, not only rational values of Kähler function but all values which differ from a rational value by a perturbation with a p-adic norm smaller than one and rationally proportional to a power of e or even its root exist p-adically in this case if they have small enough p-adic norm. The most general perturbation of the action is in the field defined by the extension of rationals defined by the root of e and algebraic numbers.

Since CP_2 action is rationally proportional to π^2, the exponent is rational if $4\pi\alpha_K$ satisfies the same condition. If the conjecture $log(p) = q_1(p)exp\left[q(p_2)\right]/\pi$ holds , then the earlier ansatz $1/\alpha_K(p) = (4/\pi)log(K^2)$ does not guarantee this, and $4/\pi$ must be replaced with a rational number $Q \simeq 4/\pi$. The presence of $log(K^2)$, K^2 product of primes, is well motivated also in this case because it gives the desired $1/\pi$ factor.

This gives for the Kähler function the expression

$$K = Q\left[q_1(p)exp\left[q_2(p)\right] + \sum_i q_1(q_i)exp\left[q_2(q_i)\right]\right]\frac{S}{S_{CP_2}} . \tag{1}$$

$exp(K)$ exists p-adically only provided that K has p-adic norm smaller than one. For given p this poses strong conditions unless one assumes that the condition $S/S_{CP_2} = p^n r$, r rational. In the case of many-particle state of CP_2 extremals this would mean that particle number is divisible by a power of p.

For single CP_2 extremal, the fact that p cannot divide $q_1(p)$ means that either Q contains a power of p or the sum of terms is proportional to a power of p. Obviously this condition is extremely strong and allows only very few primes. One might wander whether this could provide the first principle explanation for p-adic length scale hypothesis selecting primes $p \simeq 2^k$, k integer, and with prime power powers being preferred.

Since $k = 137$ (atomic length scale) and $k = 107$ (hadronic length scale) are the most important nearest p-adic neighbors of electron, one could make a free fall into number mysticism and try the replacement $4/\pi \rightarrow 137/107$. This would give $\alpha_K = 137.3237$ to be compared with $\alpha = 137.0360$: the deviation from α is .2 per cent (of course, α_K need not equal to α and the evolutions of these couplings are quite different). Thus it seems that $log(p) = q_1 exp(q_2)/\pi$ hypothesis is supported also by the properties of Kähler action and might lead to an improved understanding of the origin of the mystery prime $k = 137$. Of course, one must be extremely cautious with the numerics. For instance, one could replace 137/107 with the ratio of $137/log(M_{107}$ and in this case the M_{107} would become an "easy" prime.

2. Could infinite primes appear in the p-adicization of the exponent of

Kähler action?

The difficulties related to the p-adic continuation of Kähler function to an arbitrary p-adic number field and the fact that infinities are every day life in quantum field theory bring in mind infinite primes discussed in the chapter "Quaternions, Octonions, and Infinite Primes".

Infinite primes are not divisible by any finite prime. The simplest infinite prime is of form $\Pi = 1 + X$, $X = \prod_i p_i$, where product is over all finite primes. The factor $Y = X/(1+X)$ is in the real sense equivalent with 1. In p-adic sense it has norm $1/p$ for every prime. Thus one could multiply Kähler function by Y or its positive power in order to guarantee that the continuation to p-adic number fields exists for all primes. Of course, these states might differ physically in p-adic sense from the states having $Y = 1$. Thus it would seem that the physics of cognition could differentiate between states which are in real sense equivalent.

More general infinite primes are of form $\Pi = nX/m + n$, such that $m = \prod_i q_i$ and $n = \prod_i p_i^{n_i}$ have no common factors. The interpretation could be as a counterpart for a state of a super-symmetric theory containing fermion in each mode labelled by q_i and n_i bosons labelled in modes labelled by p_i. Also positive powers of the ratio $Y = X/\Pi$, Π some infinite prime, are possible as a multiplier of the Kähler function. In the real sense this ratio would correspond to the ratio m/n.

If this picture is correct, infinite primes would emerge naturally in the p-adicization of the theory. Since octonionic infinite primes could correspond to the states of a super-symmetric quantum field theory more or less equivalent with TGD, the presence of infinite primes could make it possible to code the quantum physical state to the vacuum functional via coupling constant renormalization.

One could also consider the possibility of defining functions like $exp(x)$ and $log(1+x)$ p-adically by replacing x with Yx without introducing the algebraic extension. The series would converge for all values of x also p-adically and would be in real sense equivalent with the function. This trick would apply to a very general class of Taylor series having rational coefficients. One could also say that p-adic physics allowing infinite primes would be very similar to real physics.

The fascination of infinite primes is that the ratios of infinite primes which are ordinary rational numbers in the real sense could code the particle number content of a super-symmetric arithmetic quantum field theory. For the octonic version of the theory natural in the TGD framework these states could represent the states of a real Universe. Universe would be an algebraic hologram in the sense that space-time points, something devoid of any structure in the standard view, could code for the quantum states of possible Universes!

The simplest manner to realize this scenario is to consider an extension of rational numbers by the multiplicative group of real units obtained from infinite primes and powers of X. Real number 1 would code everything in its structure! This group is generated as products of powers of $Y(m/n) = (m/n) \times [X/\Pi(m/n)]$ which is a unit in the real sense. Each $Y(m/n)$ would define a subgroup of units and the power of $Y(m/n)$ would code for the number of factors of a given integer with unit counted as a factor. This would give a hierarchy of integers with their p-adic norms coming as powers of p with the prime factors of m and n forming an exception and being reflected in p-adic physics of cognition, Universe would "feel" its real or imagined state with its every point, be it a point of space-time surface, of imbedding space, or of configuration space.

2.6 How to understand Riemann hypothesis

The considerations of the preceding subsection led to the requirement that the logarithmic waves $e^{iKlog(u)}$ exist in all number fields for $u = n$ (and thus for any rational value of u) implying number theoretical quantization of the scaling momenta K. Since the logarithmic waves appear also in Riemann Zeta as the basic building blocks, there is an interesting connection with Riemann hypothesis, which states that all non-trivial zeros of $\zeta(z) = \sum_n 1/n^z$ lie at the line $Re(z) = 1/2$.

I have applied two basic strategies in my attempts to understand Riemann hypothesis. Both approaches rely heavily on conformal invariance but being realized in a different manner. The universality of the scaling momentum spectrum implied by the number theoretical quantization allows to understand the relationship between these approaches.

1. First approach

In this approach (see the preprint in [16] in Los Alamos archives and the article published in Acta Mathematica Universitatis Comeniae [17]) one constructs a simple conformally invariant dynamical system for which the vanishing of Riemann Zeta at the critical line states that the coherent quantum states, which are eigen states of a generalized annihilation operator, are orthogonal to a vacuum state possessing a negative norm. This condition implies that the eigenvalues are given by the nontrivial zeros of ζ. Riemann hypothesis reduces to conformal invariance and the outcome is an analytic reductio ad absurdum argument proving Riemann hypothesis with the standards of rigor applied in theoretical physics.

2. Second approach

The basic idea is that Riemann Zeta is in some sense defined for all number fields. The basic question is what "some" could mean. Since Riemann Zeta decomposes into a product of harmonic oscillator partition functions $Z_p(z) = 1/(1-p^z)$ associated with primes p the natural guess is that $p^{1/2+iy}$ exists p-adically for the zeros of Zeta. The first guess was that for every prime p (and hence every integer n) and every zero of Zeta p^{iy} might define complex rational number (Pythagorean phase) or perhaps a complex algebraic number.

The transcendental considerations that one should try to generalize this idea: for every p and y appearing in the zero of Zeta the number p^{iy} belongs to a finite-dimensional extension of rationals involving also rational roots of e. This would imply that also the quantities n^{iy} make sense for all number fields and one can develop Zeta into a p-adic power series. Riemann Zeta would be defined for any number field in the set linearly spanned by the integer multiples of the zeros y of Zeta and it is easy to get convinced that this set is dense at the Y-axis. Zeta would therefore be defined at least in the set $X \times Y$ where X is some subset of real axis depending on the extension used.

If $log(p) = q_1 exp(q_2)/\pi$ holds true, then $y = q(y)\pi$ should hold true for the zeros of ζ. In this case one would have

$$p^{iy} = exp\left[iq(y)q_1(p)exp\left(q_2(p)\right)\right] \quad .$$

This quantity exists p-adically if the exponent has p-adic norm smaller than one. $q_1(p)$ is divisible by finite number of primes p_1 so that p^{iy} does not exist in a finite-dimensional extension of R_{p_1} unless $q(y)$ is proportional to a positive power of p_1. Also in this case the multiplication of y by the units defined by infinite primes (to be discussed later) would save the day and would be completely invisible operation in real context.

3. Logarithmic plane waves and Hilbert-Polya conjecture

Logarithmic plane waves allow also a fresh insight on how to physically understand Riemann hypothesis and the Hilbert-Polya conjecture stating that the imaginary parts of the zeros of Riemann Zeta correspond to the eigenvalues of some Hamiltonian in some Hilbert space.

a) At the critical line $Re(z) = 1/2$ (z=x+iy) the numbers $n^{-z} = n^{-1/2-iy}$ appearing in the definition of the Riemann Zeta allow an interpretation as logarithmic plane waves $\Psi_y(v) = e^{iylog(v)}v^{-1/2}$ with the scaling momentum $K = 1/2 - iy$ estimated at integer valued points $v = n$. Riemann hypothesis would follow from two facts. First, logarithmic plane waves form a complete basis equivalent with the ordinary plane wave basis from which sub-basis is selected by number theoretical quantization. Secondly, for all other powers v^k other than $v^{-1/2}$ in the denominator the norm diverges due to the contributions coming from either short $(k < -1/2)$ or long distances $(k > -1/2)$.

b) Obviously the logarithmic plane waves provide a concrete blood and flesh realization for the conjecture of Hilbert and Polya and the eigenvalues of the Hamiltonian correspond to the universal scaling momenta. Note that Hilbert-Polya realization is based on mutually orthogonal plane waves whereas the Approach 1 relies on coherent states orthogonal to the negative norm vacuum state. That eigenvalue spectra coincide follows from the universality of the number theoretical quantization conditions. The universality of the number theoretical quantization predicts that the zeros should appear in the scaling eigenvalue spectrum of any physical system obeying conformal invariance. Also the Hamiltonian generating by definition an infinitesimal time translation could act as an infinitesimal scaling.

c) The vanishing of the Riemann Zeta could code the conditions stating that the extensions involved are finite-dimensional: it would be interesting to understand this aspect more clearly.

2.6.1 Connection with the conjecture of Berry and Keating

The idea that the imaginary parts y for the zeros of Riemann zeta function correspond to eigenvalues of some Hermitian operator H is not new. Berry and Keating [23] however proposed quite recently that the Hamilton in question is super-symmetric and given by

$$H = xp - \frac{i}{2} \ . \tag{2}$$

Here the momentum operator p is defined as $p = -id/dx$ and x has non-negative real values.

H can be indeed expressed as a square $H = Q^2$ of a Hermitian super symmetry generator Q:

$$Q = \sqrt{i}\,[ix\sigma_1 + p\sigma_2] + \sqrt{\frac{i}{2}}\sigma_3 \ ,$$

$$
\begin{aligned}
\sigma_1 &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad , \\
\sigma_2 &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad , \\
\sigma_3 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad .
\end{aligned} \tag{3}
$$

By a direct calculation one finds that the following relationship holds true:

$$
Q^2 = \begin{pmatrix} xp + \frac{i}{2} & 0 \\ 0 & xp - \frac{i}{2} \end{pmatrix} \quad .
$$

The eigen spinors of Q can be written as

$$
\psi = \begin{pmatrix} u \\ v \end{pmatrix} = x^{-iy} \begin{pmatrix} x^{1/2} \\ \sqrt{\frac{y}{i}}x^{-1/2} \end{pmatrix} \quad .
$$

The eigenvalues of Q are $q = \sqrt{y}$. For $y \geq 0$ the eigenvalues are real so that Q is Hermitian when inner product is defined appropriately. Obviously y is eigenvalue of Hamiltonian.

Orthogonality requirement for the solutions of the Dirac equation requires that the inner product reduces to the inner product for plane waves $exp(iu)$, $u = log(x)$. This is achieved if inner product for spinors $\psi_i = (u_i, v_i)$ is defined as

$$
\langle \psi_1 | \psi_2 \rangle = \int_0^\infty \frac{dx}{x} \left[\overline{u}_1 v_2 + \overline{v}_1 u_2 \right] \quad . \tag{4}
$$

In the basis formed by solutions of Dirac equation this inner product is indeed positive definite as one finds by a direct calculation.

The actual spectrum assumed to give the zeros of the Riemann Zeta function however remains open without additional hypothesis. An attractive hypothesis motivated by previous considerations is that the sharpened form of Riemann hypothesis stating that n^{iy} exists for any number field provided finite-dimensional extensions are allowed for the zeros of Riemann zeta function, holds true. This implies that x^{iy} satisfies the same condition for any rational value of x. $x^{\pm 1/2}$ in turn belongs to the infinite-dimensional algebraic extension Q_C^∞ of complex rationals, when x is rational. Therefore the solutions of Dirac equation, being of form $x^{iy}x^{\pm 1/2}$, exist for all number fields for rational values of argument x.

2.6.2 Connection with arithmetic quantum field theory and quantization of time

There is also a very interesting connection with arithmetic quantum field theory and sharpened form of Riemann hypothesis. The Hamiltonian for a bosonic/fermionic arithmetic quantum field theory is given by

$$H = \sum_p log(p) a_p^\dagger a_p \ . \tag{5}$$

where $a_p^\dagger$ and a_p satisfy standard bosonic/fermionic anti-commutation relations

$$\{a_{p_1}^\dagger, a_{p_2}\}_\pm = \delta(p_1, p_2) \ . \tag{6}$$

Here $\pm$ refers to anti-commutator/commutator. The sum of Hamiltonians defines super-symmetric arithmetic QFT. The states of the bosonic QFT are in one-one correspondence with non-negative integers and the decomposition of a non-negative integer to powers or prime corresponds to the decomposition of state to many boson states corresponding to various modes p. Analogous statement holds true for fermionic QFT.

The matrix element for the time development operator $U(t) \equiv exp(iHt)$ between states $|m\rangle$ and $|n\rangle$ can be written as

$$\langle m|U(t)|n\rangle = \delta(m, n) n^{it} \ . \tag{7}$$

Same form holds true both in bosonic and fermionic QFT:s. These matrix elements are defined for all number fields allowing finite-dimensional extensions if this holds true for n^{it} so that the allowed values of t corresponds to zeros of Riemann Zeta. Similar statement holds in the case of fermionic QFT. One can say that the durations for the time evolutions are quantized in a well defined sense and allowed values of time coordinate correspond to the zeros of Riemann zeta function!

The result is very interesting from the point of view of quantum TGD since it would mean that $U(t)$ allows for the preferred values of the time parameter p-adicization ($p \ mod \ 4 = 3$) obtained by mapping the diagonal phases to their p-adic counterparts by phase preserving canonical identification. For phases this map means only the re-interpretation of the rational phase factor as a complexified p-adic number. For these quantized values of the time parameter time evolution operator of the arithmetic quantum field theory makes sense in all p-adic number fields besides complex numbers.

In the case of Berry's super-symmetric Hamiltonian the assumption that p^{iy} exists in all number fields with finite extensions allowed and the requirement that same holds true for the time evolution operator implies that allowed time durations for time evolution are given by $t = log(n)$. This means that there is nice duality between Berry's theory and arithmetic QFT. The allowed time durations (energies) in Berry's theory correspond to energies (allowed time durations) in arithmetic QFT.

2.7 Stronger variants for the sharpened form of the Riemann hypothesis

The previous form of the sharpened form of Riemann hypothesis was preceded by conjectures, which were much stronger. The strongest variant of the sharpening is that the phases p^{iy} are complex rational numbers for all primes and for all zeros ζ. A weaker form assumes that these phases belong to the square root allowing infinite-dimensional extension of rationals. Although these conjectures are probably unrealistic, they deserve a brief discussion.

2.7.1 Could the phases p^{iy} exist as complex rationals for the zeros of ζ?

The set $z = n/2 + iy$, $n > 0$ such that p^{-iy} is Pythagorean phase, is the set in which both real Riemann zeta function and the p-adic counterparts of Z_p exist for $p \ mod \ 4 = 3$. They exists also for $p \ mod \ 4 = 1$, if one *defines* $exp(ix) \equiv cos(x) + \sqrt{-1} sin(x)$: $\sqrt{-1}$ would be ordinary p-adic number for $p \ mod \ 4 = 1$. One could also allow phase factors in square root allowing algebraic extension of p-adics.

What is important that $x = 1/2$ is the smallest value of x for which the p-adic counterpart of $Z_B(p, x_p)$ exists. Already Riemann showed that the nontrivial zeros of Riemann Zeta function lie symmetrically around the line $x = 1/2$ in the interval $0 \leq x \leq 1$.

If one assumes that the zeros of Riemann zeta belong to the set at which the p-adic counterparts of Riemann zeta are defined, Riemann hypothesis follows in sharpened form.

a) Sharpened form of Riemann hypothesis does not necessarily exclude zeros with $x = 0$ or $x = 1$ as zeros of Riemann zeta unless they are explicitly excluded. It is however known that the lines $x = 0$ and $x = 1$ do not contains zeros of Riemann Zeta so that sharpened form implies also Riemann hypothesis.

b) The sharpening of the Riemann hypothesis following from p-adic considerations implies that the phases p^{iy} exist as rational complex phases for all values of $p \ mod \ 4 = 3$ when y corresponds to a zero of Riemann Zeta. Obviously the rational phases p^{iy} form a group with respect to multiplication isomorphic with the group of integers in case that y does not vanish. The same is also true for the phases corresponding to integers continuing only powers of primes $p \ mod \ 4 = 3$ phase factor.

c) A stronger form of sharpened hypothesis is that all primes p and all integers are allowed. This would mean that each zero of the Riemann Zeta would generate naturally group isomorphic with the group of integers. Pythagorean phases form a group and should contain this group as a subgroup. It might be that very simple number theoretic considerations exclude this possibility. If not, one would have infinite number of conditions on each zero of Riemann function and much sharper form of Riemann hypothesis which could fix the zeros of Riemann zeta completely:

The zeros of Riemann Zeta function lie on axis $x = 1/2$ and correspond to values of y such that the phase factor p^{iy} is rational complex number for all values of prime $p \ mod \ 4 = 3$ or perhaps even for all primes p.

Of course, the proposed condition might be quite too strong. A milder condition is that $U_p(x_p)$ is rational for single value of p only: this would mean that the zeros of Riemann Zeta would correspond to Pythagorean angles labelled by primes. One can consider also the possibility that p^{iy} is rational for all y but for some primes only and that these preferred primes correspond to the p-adic primes characterizing the effective p-adic topologies realized in the physical world.

d) If this hypothesis is correct then each zero defines a subgroup of Pythagorean phases and also zeros have a natural group structure. Pythagorean phases contain an infinite number of subgroups generated by integer powers of phase. Each such subgroup has some number N of generators such that the subgroup is generated as products of these phases. From the fact that Pythagorean phases are in a one-one correspondence with rationals, it is obvious that there exists large number of subgroups of this kind. Every zero defines infinite number of Pythagorean phases and there are infinite number of zeros. The entire group generated by the phases is in one-one correspondence with the pairs (p, y).

e) If n^{iy} are rational numbers, there must exist imbedding map f: $(n, y) \rightarrow (r, s)$ from the set of phases n^{iy} to Pythagorean phases characterized by rationals $q = r/s$:

$$(r,s) = (f_1(n,y), f_2(n,y)) \ .$$

The multiplication of Pythagorean phases corresponds to certain map g

$$\begin{aligned}
&(r_1,s_1)\circ(r_2,s_2) = [g_1(r_1,s_1;r_2,s_2),\ g_2(r_1,s_1;r_2,s_2)] \\
&= (r_1r_2 - s_1s_2, r_1s_2 + r_2s_1) \equiv (r,s)
\end{aligned}$$

such that the values of r and s associated with the product can be calculated. Thus the product operation rise to functional equations giving constraints on the functional form of the map f.

i) Multiplication of n^{iy_1} and n^{iy_2} gives rise to a condition

$$f(n,y_1)\circ f(n,y_2) = f(n,y_1+y_2) \ .$$

ii) Multiplication of n_1^{iy} and n_2^{iy} gives rise to a condition

$$f(n_1,y)\circ f(n_2,y) = f(n_1n_2,y) \ .$$

This variant of the sharpened form of the Riemann hypothesis has turned out to be un-necessarily strong. Universality Principle requires only that the real parts of the factors $p^{-x}p^{-iy}$ are rational numbers: this means that allowed phases correspond to triangles whose two sides have integer-valued length squared whereas the third side has integer-valued length.

2.7.2 Sharpened form of Riemann hypothesis and infinite-dimensional algebraic extension of rationals

The proposed variant for the sharpened form of Riemann hypothesis states that the zeros of Riemann zeta are on the line $x = 1/2$ and that p^{iy}, where p is prime, are complex rational (Pythagorean) phases for zeros. Furthermore, Riemann hypothesis is equivalent with the corresponding statement for the fermionic partition function Z_F. If the sharpened form of Riemann hypothesis holds true, the value of $Z_F(z)$ in the set of zeros $z = 1/2+iy$ of Z_F can be interpreted as a complex (vanishing) image of certain function $Z_F^\infty(1/2+iy)$ having values in the infinite-dimensional algebraic extension of rationals defined by adding the square roots of all primes to the set of rational numbers.

a) The general element q of the infinite-dimensional extension Q_C^∞ of complex rationals Q_C can be written as

$$\begin{aligned} q &= \sum_U q_U e_U \ , \\ e_U &= \prod_{i\in U} \sqrt{p}_i \ . \end{aligned} \tag{8}$$

Here q_U are complex rational numbers, U runs over the subsets of primes and e_U are the units of the algebraic extension analogous to the imaginary unit. One can map the elements of Q_C^∞ to reals by interpreting the generating units $\sqrt{p}$ as real numbers. The real images $(e_U)_R$ of e_U are thus real numbers:

$$e_U \rightarrow [e_U]_R = \prod_i \sqrt{p_i} \ .$$

b) The value of $Z_F(z)$ at $z = 1/2 + iy$ can be written as

$$Z_F(z = 1/2 + iy) = \sum_U \left[\frac{1}{e_U}\right]_R \times (e_U^2)^{-iy} \ . \tag{9}$$

Here $(e_U)_R$ means that e_U are interpreted as real numbers.

c) If one restricts the set of values of $z = 1/2+iy$ to such values of y that p^{iy} is complex rational for every value of p, then the value of $Z_F(1/2+iy)$ can be also interpreted as the real image of the value of a function $Z_F(Q_\infty|z = 1/2 + iy)$ restricted to the set of zeros of Riemann zeta and having values at Q_C^∞:

$$\begin{aligned} Z_F(1/2 + iy) &= [Z_F(Q_\infty|1/2 + iy)]_R \ , \\ Z_F(Q_\infty|1/2 + iy) &\equiv \sum_U \frac{1}{e_U} \times (e_U^2)^{-iy} \ . \end{aligned} \tag{10}$$

Note that $Z_F(Q_\infty|z = 1/2 + iy)$ cannot vanish as element of Q_∞. One can also define the Q_C^∞ valued counterparts of the partition functions $Z_F(p, 1/2+iy)$

$$\begin{aligned} Z_F(Q_\infty|1/2 + iy) &= \prod_p Z_F(Q_\infty|p, z = 1/2 + iy) \ , \\ Z_F(Q_\infty|1/2 + iy) &\equiv 1 + p^{-1/2}p^{-iy} \ , \\ Z_F(p, 1/2 + iy) &= [Z_F(Q_\infty|p, 1/2 + iy)]_R \ . \end{aligned} \tag{11}$$

$Z_F(Q_\infty|1/2+iy)$ and $Z_F(Q_\infty|p,1/2+iy)$ belong to Q_C^∞ only provided p^{iy} is Pythagorean phase.

d) The requirement that p^{iy} is rational does not yet imply Riemann hypothesis. One can however strengthen this condition. The simplest condition is that the real image of $Z_F(Q_\infty|1/2+iy)$ is complex rational number for any value of Z_F. A stronger condition is that the complex images of the functions

$$\frac{Z_F^\infty}{\prod_{p\in U} Z_p^\infty}$$

are complex rational and U is finite set of primes. The complex counterparts of these functions are given by

$$\left[\frac{Z_F^\infty}{\prod_{p\in U} Z_p^\infty}\right]_R = \frac{Z_F}{\prod_{p\in U} Z_F(p,..)} . \tag{12}$$

Obviously these conditions can be true only provided that $Z_F(1/2+iy)$ vanishes identically for allowed values of y. This implies that sharpened form of Riemann hypothesis is true. "Physically" this means that the fermionic partition function restricted to any subset of integers not divisible by some finite set of primes, has real counterpart which is complex rational valued.

2.8 Are the imaginary parts of the zeros of Zeta linearly independent of not?

Concerning the structure of the weight space of super-canonical algebra the crucial question is whether the imaginary parts of the zeros of Zeta are linearly independent or not. If they are independent, the space of conformal weights is infinite-dimensional lattice. Otherwise points of this lattice must be identified. The model of the scalar propagator identified as a suitable partition function in the super-canonical algebra for which the generators have zeros of Riemann Zeta as conformal weights demonstrates that the assumption of linear independence leads to physically unrealistic results and the the propagator does not exist mathematically for the entire super-canonical algebra. Also the findings about the distribution of zeros of Zeta favor a hypothesis about the structure of zeros implying a linear dependence.

2.8.1 Imaginary parts of non-trivial zeros as additive counterparts of primes?

The natural looking (and probably wrong) working hypothesis is that the imaginary parts y_i of the nontrivial zeros $z_i = 1/2 + y_i$, $y_i > 0$, of Riemann Zeta are linearly independent. This would mean that y_i define play the role of primes but with respect to addition instead of multiplication. If there exists no relationship of form $y_i = n2\pi + y_j$, the exponents e^{iy_i} define a multiplicative representation of the additive group, and these factors satisfy the defining condition for primeness in the conventional sense. The inverses e^{-iy_i} are analogous to the inverses of ordinary primes, and the products of the phases are analogous to rational numbers.

There would exist an algebra homomorphism from $\{y_i\}$ to ordinary primes ordered in the obvious manner and defined as the map as $y_i \leftrightarrow p_i$. The beauty of this identification would be that the hierarchies of p-adic cutoffs identifiable in terms of the p-adic length scale hierarchy and y-cutoffs identifiable in terms p-adic phase resolution (the higher the p-adic phase resolution, the higher-dimensional extension of p-adic numbers is needed) would be closely related. The identification would allow to see Riemann Zeta as a function relating two kinds of primes to each other.

A rather general assumption is that the phases p^{iy_i} are expressible as products of roots of unity and Pythagorean phases:

$$\begin{aligned} p^{iy} &= e^{i\phi_P(p,y)} \times e^{i\phi(p,y)} \quad , \\ e^{i\phi_P(p,y)} &= \frac{r^2 - s^2 + i2rs}{r^2+s^2} \quad , \quad r = r(p,y) \quad , \quad s = s(p,y) \quad , \\ e^{i\phi(p,y)} &= e^{i\frac{2\pi m}{n}} \quad , \quad m = m(p,y) \quad , \quad n = n(p,y) \quad . \end{aligned} \tag{13}$$

If the Pythagorean phases associated with two different zeros of zeta are different a linear independence over integers follows as a consequence.

Pythagorean phases form a multiplicative group having "prime" phases, which are in one-one correspondence with the squares of Gaussian primes, as its generators and Gaussian primes which are in many-to-one correspondence with primes $p_1 \; mod \; 4 = 1$. If p^{iy} is a product of algebraic phase and Pythagorean phase for any prime p, one should be able to decompose any zero y into two parts $y = y_1(p) + y_P(p)$ such that one has

$$log(p)y_1(p) = \frac{m2\pi}{n} \quad , \quad log(p)y_P(p) = \Phi_P = arctan\left[\frac{2rs}{r^2+s^2}\right] \quad . \tag{14}$$

Note that the decomposition is not unique without additional conditions. The integers appearing in the formula of course depend on p.

2.8.2 Does the space of zeros factorize to a direct sum of multiples Pythagorean prime phase angles and algebraic phase angles?

As already noticed, the linear independence of the y_i follows if the Pythagorean prime phases associated with different zeros are different. The reverse of this implication holds also true. Suppose that there are two zeros $log(p)y_{1i} = \Phi_{P_1} + q_{1i}2\pi$, $i = a, b$ and two zeros $log(p)y_{2i} = \Phi_{P_2} + q_{2i}2\pi$, $i = a, b$, where q_{ij} are rational numbers. Then the linear combinations $n_1y_{1a} + n_2y_{2a}$ and $n_1y_{1b} + n_2y_{2b}$ represent same zeros if one has $n_1/n_2 = (q_{2a} - q_{2b})/(q_{1b} - q_{1a})$.

One can of course consider the possibility that linear independence holds true only in the weaker sense that one cannot express any zero of zeta as a linear combination of other zeros. For instance, this guarantees that the super-canonical algebra generated by generators labelled by the zeros has indeed these generates as a minimal set of generating elements.

For instance, one can imagine the possibility that for any prime p a given Pythagorean phase angle $log(p)y_{P_k}$ corresponds to a set of zeros by adding to $\Phi_{P_k} = log(p)y_{P_k}$ rational multiples $q_{k,i}2\pi$ of 2π, where $Q_p(k) = \{q_{k,i}|i = 1, 2, ..\}$ is a subset of rationals so that one obtains subset $\{\Phi_{P_k} + q_{k,i}2\pi|q_{k,i} \in Q_p(k)\}$. Note that the definition of y_P involves an integer multiple of 2π which must be chosen judiciously: for instance, if y_P is taken to be minimal possible (that is in the range $(0, \pi/2)$, one obviously ends up with a contradiction. The same is true if $q_{k,i} < 1$ is assumed. Needless to say, the existence of this kind of decomposition for every prime p is extremely strong number theoretic condition.

The facts that Pythagorean phases are linearly independent and not expressible as a rational multiple of 2π imply that no zero is expressible as a linear combination of other zeros whereas the linear independence fails in a more general sense as already found. An especially interesting situation results if the set $Q_p(k)$ for given p does not depend on the Pythagorean phase so that one can write $Q_p(k) = Q_p$. In this case the set of zeros of Zeta would be obtained as a union of translates of the set Q_p by a subset of Pythagorean phase angles and approximate translational invariance realized in a statistical sense would result. Note that the Pythagorean phases need not correspond to Pythagorean prime phases: what is needed is that a multiple of the same prime phase appears only once.

An attractive interpretation for the existence of this decomposition to

Pythagorean and algebraic phases factors for every prime is in terms of the p-adic length scale evolution. The possibility to express the zeros of Zeta in an infinite number of manners labelled by primes could be seen as a number theoretic realization of the renormalization group symmetry of quantum field theories. Primes p define kind of length scale resolution and in each length scale resolution the decomposition of the phases makes sense. This assumption implies the following relationship between the phases associated with y:

$$\frac{\left[\Phi_{P(p_1)} + q(p_1)2\pi\right]}{log(p_1)} \quad = \quad \frac{\left[\Phi_{P(p_2)} + q(p_2)2\pi\right]}{log(p_2)} \quad . \tag{15}$$

In accordance with earlier number theoretical speculations, assume that $log(p_2)/log(p_1) \equiv Q(p_2, p_1)$ is rational. This condition allows to deduce how the phases p_1^{iy} transform in $p_1 \rightarrow p_2$ transformation. Let $p_1^{iy} = U_{P,p_1,y}U_{q,p_1,y}$ be the representation of p_1^{iy} as a product of Pythagorean and algebraic phases. Using the previous equation, one can write

$$p_2^{iy} = U_{P,p_2,y}U_{q,p_2,y} = U_{P,p_1,y}^{Q(p_2,p_1)}U_{q,p_1,y}^{Q(p_2,p_1)} \quad . \tag{16}$$

This means that the phases are mapped to rational powers of phases. In the case of Pythagorean phases this means that Pythagorean phase becomes a product of some Pythagorean and an algebraic phase whereas algebraic phases are mapped to algebraic phases. The requirement that the set of phases p_2^{iy} is same as the set of phases p_1^{iy} implies that the rational power $U_{P,p_1,y}^{Q(p_2,p_1)}$ is proportional to some Pythagorean phase U_{P,p_1,y_1} times algebraic phase U_q such that the the product of $U_qU_{q,p_1,y}^{Q(p_2,p_1)}$ gives an allowed algebraic phase. The map $U_{P,p_1,y} \rightarrow U_{P,p_1,y_1}$ from Pythagorean phases to Pythagorean phases induced in this manner must be one-to one must be the map between algebraic phases. Thus it seems that in principle the hypothesis might make sense.

The basic question is why the phases q^{iy} should exist p-adically in some finite-dimensional extension of R_p for every p. Obviously some function coding for the zeros of Zeta should exist p-adically. The factors $G_q = 1/(1 - q^{-iy-1/2})$ of the product representation of Zeta obviously exist if this assumption is made for every prime p but the product is not expected to converge p-adically.

Also the logarithmic derivative of Zeta codes for the zeros and can be written as

$$\frac{\zeta'}{\zeta} = -\sum_q log(q) \frac{q^{-1/2-iy}}{1-q^{-1/2-iy}} \quad . \tag{17}$$

As such this function does not exist p-adically but dividing by $log(p)$ one obtains

$$\frac{1}{log(p)} \frac{\zeta'}{\zeta} = -\sum_q Q(q,p) \frac{q^{-1/2-iy}}{1-q^{-1/2-iy}} \quad . \tag{18}$$

This function exists if the the p-adic norms rational numbers $Q(q,p)$ approach to zero for $q \to \infty$: $|Q(q,p)|_p \to 0$ for $q \to \infty$. The p-adic existence of the logarithmic derivative would thus give hopes of universal coding for the zeros of Zeta and also give strong constraints to the behavior of the factors $Q(q,p)$. The simplest guess would be $Q(q,p) \propto p^q$ for $q \to \infty$.

2.8.3 Correlation functions for the spectrum of zeros favors the factorization of the space of zeros

The idea that the imaginary parts of the zeros of Zeta are linearly independent is a very attractive but must be tested against what is known about the distribution of the zeros of Zeta.

There exists numerical evidence for the linear independence of y_i as well as for the hypothesis that the zeros correspond to a union of translates of a basic set Q_1 by subset of Pythagorean phase angles. Lu and Sridhar have studied the correlation among the zeros of ζ [30]. They consider the correlation functions for the fluctuating part of the spectral function of zeros smoothed out from a sum of delta functions to a sum of Lorentzian peaks. The correlation function between two zeros with a constant distance $K_2 - K_1 + s$ with the first zero in the interval $[K_1, K_1 + \Delta]$ and second zero in the interval $[K_2, K_2 + \Delta]$ is studied. The choice $K_1 = K_2$ assigns a correlation function for single interval at K_1 as a function of distance s between the zeros.

a) The first interesting finding, made already by Berry and Keating, is that the peaks for the negative values of the correlation function correspond to the lowest zeros of Riemann Zeta (only those contained in the interval Δ can appear as minima of correlation function). This phenomenon observed already by Berry and Keating is known as resurgence. That the anti-correlation is maximal when the distance of two zeros corresponds to a low lying zero of zeta can be understood if linear combinations of the zeros of Zeta are the least probable candidates for zeros. Stating it differently, large zeros tend to avoid the points which represent linear combinations of the smaller zeros.

b) Direct numerical support the hypothesis that the correlation function is approximately translationally invariant, which means that it depends on $K_2 - K_1 + s$ only. Correlation function is also independent of the width of the spectral window Δ. In the special $K_1 = K_2$ the finding means that correlation function does not depend at all on the position K_1 of the window and depends only on the variable s. Prophecy means that the correlation function between the interval $[K, K+\Delta]$ and its mirror image $[-K-\Delta, -K]$ is the correlation function for the interval $[2K+\Delta]$ and depends only on the variable $2K+s$ allowing to allows to deduce information about the distribution of zeros outside the range $[-K, K]$. This property obviously follows from the proposed hypothesis implying that the spectral function is a sum of translates of a basic distribution by a subset of Pythagorean prime phase angles.

This hypothesis is consistent with the properties of the the smoothed out spectral density for the zeros given by

$$\langle \rho(k) \rangle = \frac{1}{2\pi} log(\frac{k}{2\pi}) \ . \tag{19}$$

This implies that the smoothed out number of zeros y smaller than Y is given by

$$N(Y) = \frac{Y}{2\pi}(log(\frac{Y}{2\pi}) - 1) \ . \tag{20}$$

$N(Y)$ increases faster than linearly, which is consistent with the assumption that the distribution of zeros with positive imaginary part is sum over translates of a single spectral function ρ_{Q_0} for the rational multiples $q_i X_p$, $X_p = 2\pi/log(p)$, $q_i \in Q_p$, for every prime p.

If the smoothed out spectral function for $q_i \in Q_p$ is constant:

$$\rho_{Q_p} = \frac{1}{K_p 2\pi} \ , \ K_p > 0 \ , \tag{21}$$

the number $N_P(Y,p)$ of Pythagorean prime phases increases as

$$N_P(Y|p) = K_p(log(\frac{Y}{2\pi}) - 1) \ , \tag{22}$$

so that the smoothed out spectral function associated with $N_P(Y|p)$ is given by the function

$$\rho_P(k|p) = \frac{K_p}{k} \tag{23}$$

for sufficiently large values of k. Therefore the distances between subsequent zeros could quite well correspond to the same Pythagorean phase for a given p and thus should allow to deduce information about the spectral function ρ_{Q_0}. A convenient parametrization of K_p is as $K = K_{p,0}/4\pi^2$ since the points of Q_p are of form $q_i 2\pi = (n(q_i) + q_1(q_i))2\pi$, $q_1 < 1$, and $n(q_i)$ must in the average sense form an evenly spaced subset of reals.

2.8.4 Physical considerations favor the linear dependence of the zeros

The numerical evidence is at best suggestive and one can always argue that by an arbitrary small deformation of the linearly dependent zeros one obtains linearly independent zeros. This would however require that each zero of form $y_{P_i} + q2\pi$, $q \in Q_p$ is very near to a zero $\Phi_{P_{k(i,q)}} + q_{k(i,q)}2\pi$. In other words, the union of the translates of Q_p by a subset of Pythagorean phases would approximate the zeros in one-one correspondence with a larger subset of Pythagorean phases (given prime phase appears only once). This should hold for every prime and this seems rather implausible.

On the other hand, the linear dependence between zeros has deep physical implications for the basic quantum TGD, and as the following arguments demonstrate, is physically highly desirable. The precise arguments are developed later and here only the skeleton of the argument is given.

a) The zeros label the generating elements of the super-canonical algebra and the failure of the linear independence means that the weight system is not just the infinite-dimensional lattice spanned by the zeros but can be regarded as a kind of bundle like structure such that the linear combinations $log(p)y_b = \sum_{i=1}^N n_i \Phi_{P_{k_i}}$ form N-dimensional lattice, and the fiber at a given point of this lattice consists of the points $log(p)y_f = \sum_i n_i q_i 2\pi$. The set of these points is the lattice $n_1 Q_p \times n_2 Q_p \times ...$ divided by the equivalence defined by $y_{f,1} = y_{f,2}$ and for given values of n_i a discrete analog of the one-dimensional space of parallel hyper-planes of an N-dimensional defined by the equation $\sum_{i=1}^{N} n_i x^i = y$ space parameterized by the values of y. What is essential that the space of the planes is different for each point $y_b = \sum_{i=1}^{N} n_i y_{P_{k_i}}$.

b) The calculation of the scalar propagator as a partition function for the super-canonical algebra assuming linear independence gives without any restrictions to the super-canonical weights an infinite number of delta-function resonances of form $\delta(p^2 - m_n^2)$, and at the limit when all zeros of the Riemann Zeta are included in the sub-algebra of super-canonical algebra the set of delta function resonances defines a dense set on real axis. If only the super-canonical conformal weights generated by the positive zeros of Zeta are included, delta function resonances become ordinary poles of form $1/(p^2 - m_k^2)$. The resonances are infinitely narrow and form also now a dense set of real axis.

c) This result, which can be claimed to be non-physical, can be avoided if the the zeros are not linearly independent. Although the partition function cannot be calculated explicitly in this case, one can expect that the linear independence gives a reasonable first approximation and that the failure of the approximation is due to the multiple counting caused by the neglect of the fact that the planes of the fiber space can contain several equivalent points. If the zeros are linearly dependent, resonances get a finite width and singularities are avoided for real values of the masses and there are good hopes that the partition function is well-defined for the entire super-canonical algebra.

d) A further argument favoring the proposed form of zeros relates to the two hierarchies strongly suggested by quantum TGD. The first hierarchy corresponds to ordinary primes labelling p-adic length scales and corresponds to length scale resolution. The second hierarchy corresponds to a hierarchy of algebraic extensions of p-adic numbers and there is strong feeling that this hierarchy should correspond to the hierarchy of Beraha numbers $B_n = 4cos^2(\pi/n)$ associated with the phases $exp(i2\pi/n)$. The phases $exp(i\pi/p)$ or their non-trivial powers, for p prime, are even more interesting because of the structure of finite field $G(p,1)$.

One could consider the possibility that the rationals $q \in Q_p$ for any p can be ordered by their size in such a manner that this ordering corresponds to the ordering of primes with respect to size. Obviously the condition $Q_p = Q_1$ must hold true. This would imply that the products of the powers of the phases $exp(iq2\pi)$ for the lowest N values of q_i would give the Beraha phases corresponding to square free integers having corresponding primes p_i, $i = 1, ..., N$, as factors. All Beraha phases are obtained if the phases

$exp(i2\pi/p^n)$, $n = 1, 2, ..$ or their non-trivial powers, are also present. If this waves the case the full p-adic length scale hierarchy with powers of p would correspond to the hierarchy of Beraha phases. This would mean that the addition of new super-canonical conformal weights of increasing size to the sub-algebra of the super-canonical algebra would mean the increase of the dimension of the extension of p-adic numbers needed to represent the resulting phases p-adically as well as an increasing phase resolution.

e) With the assumptions about the structure of zeros of Zeta, the hierarchies defined by the subset y_{P_i} of multiples of Pythagorean prime phase angles and algebraic phases would neatly factorize and the latter would correspond to the p-adic length scale hierarchy. Pythagorean phases correspond to phases of the squares of Gaussian integers $r + is$ and the squares of Gaussian primes define naturally Pythagorean primes. The norm squared of the Gaussian prime is obviously prime: $r^2 + s^2 = p_1$, and satisfies $p_1 \ mod\ 4 = 1$. Hence there is a natural correspondence between Pythagorean prime phases and primes $p\ mod\ 4 = 1$. One can wonder whether also Pythagorean prime phase angles could be mapped to a subset of primes such that that size ordering for y_{P_i} would correspond to the size ordering for the subset of primes. As already noticed, the primeness property is actually an un-necessary strong requirement for Pythagorean phases. Needless to emphasize, these speculative assumptions would pose very strong constraints on the spectrum of zeros and are certainly testable numerically.

2.8.5 The notion of dual Zeta

These considerations lead to the idea that Riemann Zeta has a dual for which the role of multiplicative primes is taken by the additive primes. This function, call it $\zeta_d(u)$ should either vanish or diverge at points $u = p$. The partition functions for super-canonical conformal weights discussed in the chapter "Equivalence of Loop Diagrams with Tree Diagrams and Cancellation of Infinities in Quantum TGD" define analogs of Riemann Zeta involving analog of restriction of summation to integers which are products of even and odd integers and these functions indeed are singular at powers $u = p^{kx}$, $x = 2\pi k/y$, $k = 1, 2, ...$, where the transcendental values x do not depend on p. That the singularities do not occur for rational values of u is physically very satisfactory since this would mean that the scattering rates could become infinite.

The precise dual ζ_d of ζ would be the function

$$\zeta_d(u) = \sum_{\sum n(y)y, y\in Y} u^{i\sum n(y)y} = \prod_{y>0, y\in Y} \frac{1}{1-u^{iy}} \ , \tag{24}$$

where the summation is over all possible formal linear combinations of positive imaginary parts y of zeros or subset of them with non-negative coefficients $n(y)$. In the case that the zeros of Riemann Zeta are linearly independent, the set Y corresponds to all zeros. If the zeros are of the form $y = y_{P_i} + q2\pi$, $q \in Q_0$, one can restrict the consideration to a subset Y of zeros obtains by selecting only single value of $q \in Q_0$ for each y_{P_i}. The simplest option is that q is same for all values of y_{P_i}.

The interpretation as a product of bosonic partition functions defined by the zeros of ζ or subset of them, obviously makes sense, and the form of the partition function is the same as that of Riemann Zeta in the product representation. By writing $u = \rho exp(i\phi)$, $\phi \geq 0$ one finds that all terms in the product converge if the term corresponding to the smallest value $y_{min} \simeq 14.124725$ of y converges. This gives the condition $\phi > 1/y_{min} \sim 2\pi/14$. One can however extract arbitrary number of the lowest terms in the product as a separate well-defined factor and obtain a convergence above arbitrarily small $\phi_{min} = \epsilon > 0$. Thus the product is well-defined arbitrary near to real axis above it.

The limit $\phi \rightarrow 2\pi$ is well-defined and at $z = \rho e^{i2\pi}$, $\rho > 0$ the product can be written as

$$\zeta_d(\rho e^{i2\pi}) = \prod_{y\in Y} \frac{1}{1-\rho^{-2\pi y}\rho^{iy}} \ . \tag{25}$$

This expression converges to a finite result at the real axis and pole is not possible. This expression is not consistent with the requirement that $u \rightarrow 1/u$ induces a complex conjugation of ζ_d at the real axis.

The conjecture is that the limit $\phi \rightarrow 0_+$ limit of ζ_d vanishes or diverges for $u = p^{\pm 1}$. Also now the powers of $u_p = p^{kx}$ define poles of the individual factors in the product at real axis. For $u = p$ one can write

$$\zeta_d(p)\overline{\zeta}_d(p) = \prod_{y>0, y\in Y} \frac{1}{4sin^2\left[\frac{\phi(p,y)+\phi_P(y)}{2}\right]} \ . \tag{26}$$

Here U refers to the subset of zeros of Zeta. This expansion diverges for $sin^2[(\phi(p,y) + \phi_P(y))/2] < 1/4$ for sufficiently many values of y. An interesting possibility inspired by the connection with braid groups and Beraha numbers $B_n = 4cos^2(\pi/n)$ is that the numbers $4cos^2[\phi(p,y)]$ are Beraha numbers so that one would have $\phi(p,y) = \pi/n(p,y)$, $n(p,y) \geq 3$. For $n(p,y) \geq 3$ and $\phi_P(y) = 0$, all factors in the product would be larger than or equal to one so that the product would diverge. The vanishing would be thus due the Pythagorean phases. Of course, these arguments cannot be however taken completely seriously since the product expansion does not converge at the real axis.

Also the zeros $z_i = 1/2+iy_i$, $y_i > 0$, are generators of an Abelian algebra with integers $n/2+\sum_i n_i y_i$, $\sum n_i = n > 0$. The corresponding zeta function is

$$\zeta_d(u) = \prod_y \frac{1}{1-u^{-1/2-iy}} \ . \tag{27}$$

This function has even nearer resemblance to the ordinary ζ. Interestingly, the product $\prod_d \zeta_d(p)$ satisfies the identity

$$\prod_p \zeta_d(p) = \prod_y \zeta(1/2+y) \ , \tag{28}$$

if one exchanges freely the order of producting. The fact that all factors on the right hand side vanish would suggest that also $\zeta_d(p)$ vanishes for all values of p.

2.9 Why the zeros of Zeta should correspond to number theoretically allowed values of conformal weights?

The following argument provides support for the belief that the conformal weights $s = 1/2+iy$ for which $p^{1/2+iy}$ exist in a finite-dimensional extension of rationals for all values of prime p, indeed correspond to the non-trivial zeros of Zeta.

a) The basic idea of the number theoretical approach is that the conformal weights $1/2+iy$ are such that the radial waves $r^{-1/2-iy}$ exist for all rational (and thus for integer) values of r in some finite-dimensional extension of rationals. The logarithms $log(n)$ of integers can be interpreted as quantum numbers of a system defined by an arithmetic quantum field theory and Zeta function $\zeta = \sum_n n^{-iy-1/2}$ with $s = 1/2+iy$ interpreted as an inverse temperature, defines the partition function of this system.

b) On the other hand, so called Selberg's Zeta function characterizes the eigen values of the Laplacian in 2-dimensional quantum billiard systems defined in the fundamental domain of some hyperbolic subgroup G of $SL(2,Z)$

acting in the hyperbolic plane $SL(2,R)/SO(2)$ [31]. The fundamental domain is analogous to a box containing the particle. At quantum level the boundary conditions are satisfied by summing over all the G translates of $SL(2,R)$ invariant Green function with respect to the second argument. Physically this is analogous to putting to all copies of the fundamental domain an image charge. The confinement to the fundamental domain selects from the continuous energy spectrum a discrete sub-spectrum. Selberg's Zeta (its logarithmic derivative) has the allowed energy eigen values as its zeros (poles). Furthermore, the energy eigen values of Laplacian are of form $E = -l(l+1)$, where $l = -1/2 - iy$ is identifiable as the counterpart of conformal weight and has the same form as the zeros of Zeta. y has discrete spectrum of values characterized by the choice of G. The density of the energy eigenvalues is amazingly similar to that of Zeta.

c) On basis of above resemblances one can argue that Riemann Zeta (its logarithmic derivative) characterizes the purely number theoretical spectrum as its zeros (poles). If this is the case, the zeros of Zeta would coincide with the number theoretically allowed conformal weights $1/2 + iy$.

2.9.1 The p-adically existing conformal weights are zeros of Zeta for 1-dimensional systems allowing discrete scaling invariance

The obvious question is whether one could reduce number theory to symmetry. The following considerations suggests that $D \geq 2$-dimensional spaces do not allow a system having zeros of Zeta as its spectrum.

a) The density of states of the Selberg Zeta function differs in some aspects from that of Zeta so that Riemann Zeta probably has no interpretation as a Selberg Zeta function of a number theoretical system. For instance, the average density of states with respect to y grows linearly rather than logarithmically although the fluctuating part of the density of states is formally very similar to that of Zeta.

b) Lobatchevski space (the hyperboloid of the 4-dimensional future light cone) has $SL(2,C)$ as its isometry group. The energy spectrum of Laplacian in this case is of the form $E = -l(l+2) = 1 + y^2$ with $l = -1 - iy$ and thus different from the spectrum of 2-dimensional case and of Riemann Zeta. Due to the higher dimension of the system the mean density of states grows even faster than in the 2-dimensional case so that there seems to be no hope of getting the density of states of Riemann Zeta.

Only one-dimensional systems give hopes of the required logarithmically varying mean density of states. The simplest candidate one can imagine is a system with discrete scaling invariance.

a) Instead of Laplacian, and in complete accordance with the view that conformal invariance is the key to the understanding of Riemann Zeta, one can consider the scaling operator $L_0 = xd/dx$ acting at the half line R_+ so that the Green functions defined by the equation

$$(L_0 + z)G(x, x_1) = (xd/dx + z)G(x, x_1) = \delta(\frac{x}{x_1} - 1) \quad (29)$$

become the object of interest. The solution can be written as

$$G(x, x_1|z) = (\frac{x}{x_1})^z \times \theta(\frac{x}{x_1} - 1) \ . \quad (30)$$

Here $\theta(x)$ denotes the step function. The requirement that the integrals

$$\int \overline{G}(x, x_1|z_1)G(x, x_1|z_2)dx$$

reduce to the inner products of ordinary plane waves when $ln(x/y)$ is taken as an integration variable forces the condition $z = 1/2 + iy$. In fact, this might be seen as the physicist's "proof" of the Riemann hypothesis.

b) Following the construction of the automorphic Green functions in the hyperbolic plane described in [31], the next step is to form a sum over the $x-$ scaling transforms of $G(x, x_1|z)$ by summing over the integer scaled values nx of x to form a well defined Green function in the fundamental domain associated with the semigroup of integer scalings. Any interval $[n, 2n]$ forms a fundamental domain. This gives

$$\begin{aligned} G_I(x, x_1|\frac{1}{2} + iy) &= \sum_n G(nx, x_1|\frac{1}{2} + iy) = \sum_n (\frac{nx}{x_1})^{\frac{1}{2}+iy} \\ &= \zeta(\frac{1}{2} + iy) \times (\frac{x}{x_1})^{\frac{1}{2}+iy} \ . \end{aligned} \quad (31)$$

The resulting Green function is proportional to Riemann Zeta at the critical line and vanishes for the zeros of Zeta. Note that the logarithmic derivative of ζ divided by $log(p)$ exists in a finite-dimensional extension of R_p for $x = n/2 + i\sum_k m_k y_k$ if the basic number theoretical requirements on the phases p^{iy} defined by the zeros of Zeta are satisfied: in particular $log(p_1)/log(p)$

must have R_p norm which approaches zero for larger values of p_1. Hence the logarithmic derivative of Zeta could codes the number theoretical physics universally.

c) In the usual approach [31] the integral of G_I over the fundamental domain would give the density of states $d(E)$. In the recent case the integration over the fundamental domain $[1,2]$ gives just ζ function

$$\int_1^2 G_I(x,x|-\frac{1}{2}+iy)dx \quad = \quad \sum_n n^{-\frac{1}{2}-iy} = \zeta(\frac{1}{2}+iy) \ . \tag{32}$$

The interpretation as a density of states is obviously not possible. The proof for the Riemann hypothesis to be discussed later allows to interpret the vanishing of Riemann Zeta as as orthogonality of physical states labelled by zeros of Zeta with a tachyonic vacuum state with a vanishing conformal weight. The vanishing of Green function could also now have an interpretation stating that the physical states labelled by non-trivial zeros are orthogonal to the scaling invariant tachyonic vacuum.

d) Quite generally, the imaginary part of the logarithmic derivative of any real function $f(E)$ for which energy eigenvalues E_n correspond to zeros of unit multiplicity, defines the density of states as a sum over delta functions. $G(y) = \zeta(1/2+iy)$ is real at the critical line as is also its logarithmic derivative apart from delta function singularities of the imaginary part at the zeros of Zeta so that its logarithmic derivative indeed gives the density of zeros of Zeta:

$$d(y) \quad = \quad \frac{1}{\pi}Im\left[i\frac{dlog\left[\zeta(\frac{1}{2}+iy)\right]}{dy}\right] = \sum_n \delta(y-y_n) \ . \tag{33}$$

This ultra simple model realizes the idea that the logarithmic derivative of Green function naturally associated with a system invariant under the semi-group of integer scalings codes as its poles the zeros of Zeta. The p-adic existence of the Green function in turn is equivalent with the requirement that the spectrum corresponds to the zeros of Zeta.

2.9.2 Realization of discrete scaling invariance as discrete 2-dimensional Lorentz invariance

Both the role of the hyperbolic groups and the fact that in quantum TGD zeros of Zeta label representations of Lorentz group, encourage to think that the 1-dimensional hyperbolic subspace $t^2 - x^2 = constant$ of 2-dimensional Minkowski space having Lorentz group $SO(1,1)$ as its symmetries realizes the above described system physically. The counterpart of the hyperbolic subgroup G of $SL(2,R)$ would the semigroup of Lorentz transformations defining integer scalings of the second light like coordinate:

$$u \equiv t + z \rightarrow nu \ \ , \ \ v \equiv t - z \rightarrow \tfrac{1}{n}v \ \ .$$

This semigroup corresponds to the diagonal semi-subgroup of $SL(2,Q)$ consisting of matrices $diag(\lambda, 1/lambda) = diag(n, 1/n)$. The reduction to semigroup is natural by the presence of the p-adic length scale cutoff unavoidable in p-adicization.

Taking $u = t + z$ as the coordinate of the hyperboloid, the situation reduces to that already considered. Infinitesimal Lorentz boost acts as a scaling operator and its eigenvalues correspond to the zeros of Zeta by number theoretic existence requirements. The matrices $diag(p, 1/p)$, p prime, are completely analogous to the group elements g_0 defining primitive periodic orbits in the higher-dimensional case so that prime numbers are naturally realized as discrete Lorentz transformations. Prime Lorentz transformations and their inverses generate rational Lorentz group. The length of the primitive periodic orbit corresponds to the scaling parameter $log(p)$ defining the scaling by p as an exponentiated scaling transformation $u \rightarrow exp(log(p))u = pu$.

3 Universality Principle and Riemann hypothesis

The basic definition of $\zeta(s = x + iy)$ based on the product formula does not converge for $Re[s] \leq 1$. One can however define 'universal' ζ, call it $\hat{\zeta}$, as the product of the partition functions $Z_{p_1}(s) = 1/(1 - p^{-s})$, in the subset of complex plane, where the factors Z_{p_i} are complex algebraic numbers. The idea is to regard the value of $\hat{\zeta}$ as an element of an infinite-dimensional algebraic extension of the rationals containing all roots of primes. $\hat{\zeta}$ can be regarded as a vector with infinite number of components and is completely well defined despite the fact that the product expansion does not converge as an ordinary complex number unless one somehow specifies how the 'producting' is done.

In case that the factors $|Z_{p_1}|^2$ of the partition functions $Z_{p_1} = 1/(1-p^{-z})$ are complex rationals, one can rewrite the product formula by applying adelic formula to the norm squared $|Z_{p_1}|^2$ appearing in the product formula. The basic hypothesis is that the product of the p-adic norms of the complex norm squared of the function $\hat{\zeta}$ defined by the product formula obtained by changing the order of producting gives the norm squared of the analytically continued ζ in the region $(Re[s] < 1 \ , \ Im[s] \neq 0)$ at the points, where the factors $|Z_{p_1}|^2$ are algebraic numbers: $|\hat{\zeta}|^2 = \prod_p N_p(|\hat{\zeta}|^2) = |\zeta|^2$. A milder version of this hypothesis is that the product of the p-adic norms squared of $|\hat{\zeta}|^2$ converges to some function proportional to $|\zeta|^2$.

If this hypothesis is correct, the following vision giving good hopes about the proof of the Riemann hypothesis, suggests itself.

a) $|\hat{\zeta}|^2$ is a number in an infinite-dimensional algebraic extension of rationals and can vanish only if it contains a rational factor which vanishes. The vanishing of this factor is possible if it is a product of an infinite number of moduli squared $|Z_{p_1}(z)|^2$ having a rational value. For the values of y for which this is true on the line $Re[s] = n + 1/2$ correspond to the phases p_1^{-iy} having the following general form.

$$
\begin{aligned}
&p^{-iy} = U_1 U = \frac{(r_1 + is_1\sqrt{k(p_1,y)})}{\sqrt{p_1}} \times \frac{(r + is\sqrt{k(p_1,y)})}{n_1} \quad , \\
&r_1^2 + s_1^2 k(p_1,y) = p_1 \quad , \\
&r^2 + s^2 k(p_1,y) = n_1^2 \quad .
\end{aligned}
$$

$r_1^2 + s_1^2 k(p_1,y) = p_1$ condition is solved by $k(p_1,y) = \sqrt{p_1 - m^2}$, $m < \sqrt{p}$. $r^2 + s^2 k(p_1,y) = n_1^2$ condition is satisfied if U is a product of even powers of the phases of type U_1. Unless $k(p_1,y)$ is not square, the phases correspond to orthogonal triangles with one short side having integer valued length and the other sides having integer valued length squared.

b) If y defines rational value of $|Z_{p_1}(z)|^2$, also its integer multiples ny do the same. If the values of integers $k(p_1,y)$ do not depend on the value of y, the allowed values of y generate an additive group having integers as a coefficient ring. Even powers of the phases guaranteing the rationality of $|Z_{p_1}(z)|^2$ on the line $Re[s] = 1/2$, guarantee rationality on the lines $Re[s] = n$.

c) Especially important subset of these phases correspond to the choice $k_1 = 1$. These phases correspond to Gaussian primes having the form $G = r_1 + is_1$, $r_1^2 + s_1^2 = p_1$, $p_1 \ mod \ 4 = 1$, and can compensate the irrationality of the $p_1^{-n-1/2}$ factor only in this case. The products of the squares of Gaussian primes define Pythagorean triangles and the corresponding phases are rational. Rather interestingly, the linear superpositions $y = n_1 y_2 + n_2 y_2$ of only *two* Pythagorean values of y_i form a dense subset of reals. Eisenstein primes having the general form $r_1 + s_1 w$, $w = -1/2 \pm \sqrt{3/2}$, $r_1^2 + s_1^2 - r_1 s_1 = p_1$, $p_1 \ mod \ 3 = 1$, are second, probably very important class of complex primes. They can compensate the irrationality of the $p_1^{-n-1/2}$ factor for

p_1 mod $3 = 1$ (note that the 1/2 is not relevant for the phase). Also other phases are needed since for primes satisfying p_1 mod $4 = 3$ and p_1 mod $3 = 2$ simultaneously neither Gaussian nor Eisenstein primes can compensate the irrationality of the $p_1^{-1/2} p_1^{-iy}$ factor.

d) The lines on which the real parts for an infinite number of factors Z_{p_1} can be rational, correspond to the lines $Re[s] = n/2$. This in turns leads to the conclusion that the norm squared of $\hat{\zeta}$ can vanish only on the lines $Re[s] = n/2$. If the norm squared of the $\hat{\zeta}$ coincides with the norm squared of the analytically continued ζ, Riemann hypothesis follows since it is known that the lines $Re[s] = n/2$, $n \neq 1$ do not contain zeros of ζ.

In the following this vision is developed in detail and it is shown that it survives the basic tests.

3.1 Detailed realization of the Universality Principle

Universality Principle states that ζ vanishes only if $|\hat{\zeta}|^2$ understood as a number in an infinite-dimensional algebraic extension of rationals vanishes and hence must contain a rational factor resulting from an infinite number of rational factors Z_{p_1}. This hypothesis alone makes Riemann hypothesis very plausible. In this section an attempt to reduce the Universality Principle to something more concrete is made. Adelic formula and the hypothesis that the norm of $|\hat{\zeta}|^2$ defined by the modified adelic formula equals to $|\zeta|^2$ are described and shown to imply Universality Principle if the modified adelic formula defines a norm in the infinite-dimensional algebraic extension of rationals. The conditions guaranteing the rationality and the reduction of the p-adic norm of $|Z_{p_1}|^2$ are derived, and the connection between Pythagorean phases and basic facts about Gaussian and Eisenstein primes are summarized.

3.1.1 Modified adelic formula and Universality Principle

Although the product representation of ζ does not converge absolutely for $Re[s] \leq 1$, one can consider the possibility that the convergence of the function $\hat{\zeta}$ defined by the product representation occurs in some exceptional points in some natural sense. The points at which the value of $\hat{\zeta}$ belongs to the infinite-dimensional algebraic extension of rationals are obviously excellent candidates for these points. $\hat{\zeta}$ identified as an element of this algebraic extension certainly exists mathematically as a vector with an infinite number of components. The convergence in the strong sense would mean that the interpretation of the algebraic numbers of the algebraic extension as real numbers in the expression of $\hat{\zeta}$ gives the analytically continued ζ somehow. In the weak sense the convergence would mean that the complex norm squared for $\hat{\zeta}$, if defined in a suitable sense, equals or is proportional, to the norm squared of the analytically continued ζ.

1. Modified Adelic formula and Universality Principle

The fact that the product formula for ζ at rational points converges only conditionally, suggests that one should be able to device a natural method of 'producting' giving rise to the norm squared of the analytically continued ζ. Adelic formula provides very attractive approach to this problem (the appearance of the norm squared instead of norm is motivated by the Adelic formula).

The adelic formula expresses the real norm of a rational number as a product of the inverses of the p-adic norms

$$\frac{1}{|x|_R} = \prod_p |x|_p \ . \tag{34}$$

This formula generalizes also to the norms of the complex rationals. How to generalize this formula to the infinite-dimensional algebraic extension of rationals? The simplest possibility is to write the complex norm squared as vector in the infinite-dimensional extension having rational coefficients and to apply adelic formula to each factor separately.

$$\begin{aligned} |x|_R &= \sum_k e_R^{k)} \prod_p |\frac{1}{x_k}|_p \ , \\ |x| &= \sum_k e^{k)} x_k \ . \end{aligned} \tag{35}$$

Here $e^{k)}$ denote the units of the infinite-dimensional algebraic extension (products of roots of primes and analogous to imaginary unit) and $e_R^{k)}$ denote the evaluations of these units identified as real numbers. The resulting norm is indeed equal to the real norm when the resulting number is interpreted as a real number.

In the case that the factors Z_{p_1} of ζ are complex rationals, one can write the real norm of the real ζ for $Re[s] > 1$ as a product

$$|\zeta(z)|^2 = \ = \prod_{p_1} \left[\prod_p N_p(|\frac{1}{Z_{p_1}(z)}|^2) \right] \equiv \prod_{p_1} \left[\prod_p N_p(|Z_{p_1}^{p)}(z)|^2) \right] \ . \tag{36}$$

Here $N_p(x)$ denotes the p-adic norm of number x. This formula explains why one must define the p-adic zeta as an arithmetic inverse of the real ζ. The generalization of this formula to the case that $\hat{\zeta}^2$ has values in the set of the complex rationals is straightforward.

The problem with this representation is that the product over primes p_1 does not converge in an absolute sense for $Re[s] \leq 1$. By a suitable rearrangement of a conditionally convergent product a convergence to any number can be achieved. This suggests that one could find some unique manner to rearrange the terms to a convergent expression converging to $|\zeta|^2$. A unique definition indeed suggests itself: the analytic continuation of ζ from the region $Re[s] > 1$ might be equivalent with the exchange of the order of 'producting' in the expression of ζ:

$$\begin{aligned} |\hat{\zeta}(z)|^2 &= \prod_p N_p(|\frac{1}{\zeta(z)}|^2 = \prod_p \left[\prod_{p_1} N_p(|\frac{1}{Z_{p_1}(z)}|)\right] \\ &= \prod_p N_p(|\frac{1}{\zeta}|^2) = \prod_p N_p(|\zeta^{p)}|)^2 \ . \end{aligned} \tag{37}$$

The minimal working hypothesis is that $|\hat{\zeta}|^2$ defined as the product its p-adic norms equals to $|\zeta|^2$ at points, where its values are *rational*:

$$\prod_p N_p(|\hat{\zeta}|^2) = |\zeta|^2 \ . \tag{38}$$

The generalization to the algebraic extension of rationals is straightforward since the p-adic norm squared is sum over the p-adic norms of the components of the algebraic extension with various units $e^{k)}$ of the algebraic extension multiplying them interpreted as real numbers $e_R^{k)}$

$$\begin{aligned} \prod_p N_p(|\hat{\zeta}|^2) &= \sum_k e_R^{k)} \prod_p N_p(\frac{1}{|\hat{\zeta}|_k^2}) = |\zeta|^2 \ , \\ |\hat{\zeta}|^2 &= \sum_k e^{k)} |\zeta|_k^2 \ . \end{aligned} \tag{39}$$

From this formula Universality Principle follows automatically. Since $|\hat{\zeta}|^2$ can be regarded as a vector having infinite number of components, the only manner to achieve the vanishing of $\prod_p N_p(|\hat{\zeta}|^2)$ is to require that it contains a vanishing rational factor. As will be found, the points at which infinite number of the factors of $|\hat{\zeta}|^2$ can be rational, very probably belong to the lines $Re(s) = n/2$. Thus the Universality Principle, and as it seems, also Riemann hypothesis, reduces to the statement that the modified Adelic formula defines a genuine norm which vanishes only when the vector is a null vector and is equal to $|\zeta|^2$. Of course, one could consider also the possibility that this norm is proportional to $|\zeta|^2$.

3.1.2 The conditions guaranteeing the rationality of the factors $|Z_{p_1}|^2$

Universality Principle states that zeros of ζ correspond to zeros of $|\hat{\zeta}|^2$. This quantity, when well-defined, belongs to an infinite-dimensional real algebraic extension of rationals, and its vanishing is possible if it contains a vanishing rational factor which is product of an infinite number of factors Z_{p_1} which are rational. $|\hat{\zeta}|^2$ is the product of the factors

$$\frac{1}{Z_{p_1}(x+iy)Z_{p_1}(x-iy)} = 1 - 2p_1^{-x}Re[p_1^{iy}] + p_1^{-2x} \quad . \tag{40}$$

This expression equals to a rational number q, if one has

$$Re[p_1^{iy}] = \frac{qp_1^{x} - p_1^{-x}}{2} \quad . \tag{41}$$

In this case the integer multiples ny do not satisfy the rationality condition, to say nothing about the superpositions of different values of y. It is also implausible that this condition would hold true for an infinite number of primes p_1 required by the vanishing of a rational factor of $\hat{\zeta}$.

An alternative manner to achieve rationality is by requiring that the two terms are separately rational. p_1^{-2x} factor is rational only if one has $x = n/2$. To achieve rationality $Re[p_1^{iy}]$ should contain a factor compensating the irrationality of the $p_1^{-n/2}$ factor somehow. On the lines $Re[s] = x = n/2$ one has

$$\frac{1}{Z_{p_1}(n/2+iy)Z_{p_1}(n/2-iy)} = 1 - 2p_1^{-n/2}Re[p_1^{iy}] + p_1^{-n} \quad .$$

It is of crucial importance that the moduli squared depend on the real part of p_1^{iy} only. If this is rational, rationality is achieved for even values of n.

On the lines $Re[s] = n + 1/2$ rationality is achieved provided that p_1^{iy} factors contain the phase factor $(r_1 + is_1\sqrt{k})/\sqrt{p_1}$ compensating the $p_1^{-1/2}$ factor and multiplying a factor which of the same type:

$$\begin{aligned} p_1^{iy} &= U_1 U = \frac{(r_1 + is_1\sqrt{k})}{\sqrt{p_1}} \times \frac{(r + is\sqrt{k})^2}{r^2 + s^2 k} \ , \\ r_1^2 + s_1^2 k_1 &= p_1 \ . \end{aligned} \tag{42}$$

The latter equation is satisfied if one has

$$k = \sqrt{p_1 - m^2} \ , \ 0 < m < \sqrt{p} \ . \tag{43}$$

On the lines $Re[s] = n$ one must have

$$p_1^{iy} = \frac{(r + is\sqrt{k})^2}{r^2 + s^2 k} \ . \tag{44}$$

The overall conclusions are following.

a) The vanishing of $|\hat{\zeta}|^2$ requires only the rationality of the *real parts* of Z_{p_1} for infinite number of values of p_1. The basic ansatz allows rationality only on the lines $Re[s] = n/2$ and my subjective feeling is that it is extremely implausible that exceptional ansatz gives rise to the rationality of an infinite number of $|Z_{p_1}|^2$ factors. That this is really the case might turn out to be difficult part in attempts to prove Riemann hypothesis even if one has proved the identity $\prod_p N_p(|\hat{\zeta}|^2) = |\zeta|^2$ and that this product defines a norm.

b) Rationality requirement allows p_1^{-iy} to consist of the products of the phases of very general algebraic numbers $r + is\sqrt{k}$. The products of these numbers are always of same form and their norm squared is $r^2 + s^2 k$. Geometrically these numbers correspond to orthogonal triangles with one or two sides having integer valued length and remaining side having integer valued length squared.

c) For given value of y all integer multiples ny of y provide a solution of the rationality conditions. It is not necessary to require that the algebraic extensions $r + is\sqrt{k(p_1, y_i)}$ associated with y_1 and y_2 satisfying the condition, are same for given value of p_1: that is, one can have

$$k(p_1, y_1) \neq k(p_1, y_2) \ .$$

For $k(p_1,y_1)=k(p_1,y_2)$ also the linear combinations $m_1y_1+n_1y_2$ satisfy rationality conditions. For the minimal solution to the rationality conditions, only multiples of each y solve the rationality conditions. For the maximal solution all solutions y_i correspond to the same algebraic extension for given p_1 and unrestricted linear superposition of the y_i holds true.

d) For $p\ mod\ 4=1$ rational phase factors p_1^{-iy} defined by the powers of the Gaussian primes provide the minimal manner to achieve rationality such that unrestricted superposition of solutions holds true. For $p_1\ mod\ 4=3$ and $p_1\ mod\ 3=1$ the minimal manner to achieve compensation is by using Eisenstein primes. For the primes $p_1\ mod\ 4=3$ and $p_1\ mod\ 3=1$ one cannot compensate $\sqrt{p_1}$ factor using Gaussian or Eisenstein primes and a more general algebraic extension of integers is necessary. For given prime p_1 there is finite number of possible algebraic extensions.

3.1.3 The conditions guaranteing the reduction of the p-adic norm

The term p_1^{-iy} appearing in the factors $1/Z_{p_1}$ is inversely proportional to integers and thus have p-adic norm which is larger than one for the primes appearing as factors of the integer n_1. Some mechanism guaranteing the reduction of the p-adic norm must be at work and this mechanism gives strong conditions on the allowed phases p_1^{iy}.

The condition guaranteing the reduction is very general. What is required is the reduction of the p-adic norm

$$|X\overline{X}|_p \ , \quad X=1-Up_1^{iy} \ , \quad U=(\epsilon p_1)^{-n/2} \ . \tag{45}$$

Here one has $\epsilon=1$ for even values of n whereas for for odd values of n one has $\epsilon=\pm 1$ depending on whether the square root exists or not p-adically: the sole purpose of this factor is to take care that the p-adic counterpart of U is an ordinary p-adic number.

By writing

$$p_1^{-iy}\equiv cos(\phi)+isin(\phi) \ ,$$

one obtains

$$|X\overline{X}|_p=|1-2Ucos(\phi)+U^2|_p \ .$$

Not surprisingly, the vanishing of the norm modulo p implies in modulo p accuracy

$$U = cos(\phi) - \sqrt{-1}sin(\phi) \ .$$

Since U must be real, the only possible manner to satisfy the condition is to require that

$$sin(\phi) = 0 \ mod \ p \ , \quad cos(\phi) = 1 \ mod \ p \ . \tag{46}$$

Clearly, ϕ must correspond to angle 0 or π in modulo p accuracy. What this condition says is that partition functions Z_{p_1} are real in order p. This is very natural condition on the line $Re[s] = 1/2$ where the ζ is indeed real.

The condition $cos^2(\phi) = 1 \ mod \ p$ implies

$$p_1^n \ mod \ p = 1 \ . \tag{47}$$

p_1 can be always written as a power $p_1 = a^k$ of a primitive root a satisfying $a^{p-1} = 1$ modulo p such that k divides $p-1$. Thus $p_1^n \ mod \ p = 1$ holds true only only if $n \ mod \ (p-1)/k = 0$ is satisfied.

The conditions guaranteing modulo p reality of Z_{p_1} for prime p dividing the denominator of p_1^{-iy}, when written explicitly, give

$$\begin{array}{lll} Re[s] = n \ : & r^2 - s^2k = r^2 + s^2k \ , & \frac{2rs}{r^2+s^2k} = 0 \ , \\ Re[s] = n + \frac{1}{2} \ : & (r^2 - s^2k)r_1 - 2rss_1k = r^2 + s^2k \ , & \frac{2rsr_1 + (r^2-s^2k)s_1}{r^2+s^2k} = 0 \ . \end{array} \tag{48}$$

In the case of Gaussian primes ($k = 1$) also second option is possible since the multiplication with $\pm i$ yields new rational phase factor: this option corresponds simply the exchange of $r^2 - s^2$ and $2rs$ factors in the formula above.

Rather general solution to the conditions can be written rather immediately. In both cases the conditions

$$s \ mod \ p^2 = 0 \ , \quad r \ mod \ p = 0 \tag{49}$$

are satisfied. Note that $s \ mod \ p^2 = 0$ is necessary since $r^2 + s^2k \ mod \ p = 0$ holds true. Besides this the conditions

$$
\begin{aligned}
&r_1^2+s_1^2 k \; mod \; p = 1 && \text{for } Re[s]=n \ , \\
&s_1 \; mod \; p = 0 \ \& \ r_1 \; mod \; p = 1 && \text{for } Re[s]=n+\tfrac{1}{2} \ ,
\end{aligned}
\tag{50}
$$

are satisfied.

If p_1^{-iy} is inversely proportional to integer containing as factors powers of a prime p larger than p_1, the reduction of the norm cannot occur for $Re[s]=1/2$ but is possible for sufficiently large values of $Re[s]=n/2$. For $p_1=2$ and $p_1=3$ factors the reduction of the norm is certainly not possible on the line $Re[s]=1/2$ since the condition $2p+1\leq p_1$ cannot be satisfied for any prime in these cases. The reduction of the p-adic norm of the ζ suggests strongly that the condition $2p_i+1\leq p_1$ is satisfied for large primes p_1 and odd primes p_i. The condition is satisfied always for $p_i=2$ and $p_1\geq 3$. If it is satisfied completely generally, the phase factors associated with Z_3 must be of the general form

$$
3^{-iy} = \frac{(\pm 1\pm\sqrt{2}i)}{\sqrt{3}}\times\frac{(r(y)+i\sqrt{2}s(y))^2}{r^2(y)+2s^2(y)} \ , \quad r^2(y)+2s^2(y)=3^k \text{ or } 2\times 3^k \ .
$$

This condition and similar conditions associated with larger primes give very strong constraints on the zeros.

The general conclusions are following.

a) The reduction of the p-adic norm and the related modulo p reality of Z_{p_1} is the p-adic counterpart for the reality of ζ on the critical line which suggests that it might occur completely generally. It requires that $p_1^n \; mod \; p=1$ holds true for all primes appearing as factors of the denominator n_1 of the rational part of the phase p_1^{-iy}.

b) If the denominator of p_1^{-iy} is square-free integer, the p-adic norm of Z_{p_1} is never larger than unity except possibly in the diagonal case $p=p_1$.

c) In the diagonal case the norm grows like p_1^{n+1} for $Re[s]=n+1/2$ and p_1^n for $Re[s]=n$. This conforms with the fact that ζ has no zeros for $Re[s]\geq 1$ but has zeros for $Re[s]=-2n$.

d) If rational points of ζ obey linear superposition, then the rational points on the lines $Re[s]=n$ contain an even number of y_i:s needed to achieve the rationality of $Re[p^{-iy}]$. Hence the denominator tends to have larger p-adic norm than it can have on the line $Re[s]=1/2$. This means that the line $Re[s]=1/2$ is optimal as far as zeros of $|\hat{\zeta}|^2$ are considered. It can however happen that in the product $p_1^{iy_1}p_1^{iy_2}$ complex conjugates of factor phases can compensate each other so that the p-adic norm of $p_1^{i(y_1+y_2)}$

is not always larger than the norms of the factors. In particular, the factors $(r_1 + is_1\sqrt{k})/\sqrt{p_1}$ could cancel in the product $p_1^{iy_1}p_1^{-iy_2}$ This mechanism could imply the emergence small values of ζ for $y_{ij} = y_i - y_j$ on the line $Re[s] = 1$ required by the inner product property of the Hermitian form defined by the super-conformal model for the zeros of ζ.

3.1.4 Gaussian primes and Eisenstein primes

The general manner to satisfy the rationality requirement is to assume that the phases p_1^{iy} correspond to orthogonal triangles with one or two sides with an integer valued length and one side with integer valued length squared. A rather general and mathematically highly interesting manner to realize the rationality of the the phases $p_1^{-n/2}p_1^{iy}$ is by choosing the phases to be products of Gaussian or Eisenstein primes [18].

Gaussian primes consist of complex integers $e_i \in \{\pm 1, \pm i+\}$, ordinary primes $p \ mod \ 4 = 3$ multiplied by the units e_i to give four different primes, and complex Gaussian primes $r \pm is$ multiplied by the units e_i to give 8 primes with the same modulus squared equal to prime $p \ mod \ 4 = 1$. Every prime $p \ mod \ 4 = 1$ gives rise to 8 non-degenerate Gaussian primes. Pythagorean phases correspond to the phases of the squares of complex Gaussian integers $m+in$ expressible as products of even powers of Gaussian primes $G_p = r+is$:

$$G_p = r + is \ \ , \ \ G\overline{G} = r^2 + s^2 = p \ \ , \ \ p \ \text{prime} \ \& \ \ p \ \ mod \ \ 4 = 1 \ \ . \tag{51}$$

The general expression of a Pythagorean phase expressible as a product of even number of Gaussian primes is

$$U = \frac{r^2 - s^2 + i2rs}{r^2 + s^2} \ \ . \tag{52}$$

By multiplying this expression by a Gaussian prime i, one obtains second type of Pythagorean phase

$$U = \frac{2rs + i(r^2 - s^2)}{r^2 + s^2} \ \ . \tag{53}$$

Gaussian primes allow to achieve rationality of $p_1^{-n+1/2}p^{-iy}$ factor for $p_1 \ mod \ 4 = 1$. The generality of the mechanism suggests that Gaussian primes should be very important. For $Re[s] \neq n/2$ it is not possible to achieve complex rationality with any decomposition of p_1^{iy} to Gaussian primes.

Besides Gaussian primes also so called Eisenstein primes are known to exist [18] and the fact that only the rationality of the real parts of $1/Z_{p_1}$ factors is necessary for the rationality of $|Z_{p_1}|^2$ means that they are also possible. Note however that now the multiplication the phase by $\pm i$ makes the real part of the phase irrational, and is thus not allowed. Thus only four-fold degeneracy is present now for ζ.

Whereas Gaussian primes rely on modulo 4 arithmetics for primes, Eisenstein primes rely on modulo 3 arithmetics. Let $w = exp(i\phi)$, $\phi = \pm 2\pi/3$, denote a nontrivial third root of unity. The number 1-w and its associates obtained by multiplying this number by ± 1 and $\pm i$; the rational primes $p \ mod \ 3 = 2$ and its associates; and the factors $r + sw$ of primes $p \ mod \ 3 = 1$ together with their associates, are Eisenstein primes. One can write Eisenstein prime in the form

$$w = r - \frac{s}{2} + is\frac{\sqrt{3}}{2} \ . \tag{54}$$

What might be called Eisenstein triangles correspond to the products of powers of the squares of Eisenstein primes and have integer-valued long side. The sides of the orthogonal triangle associated with a square of Eisenstein prime E_p have lengths

$$(r^2 - rs - \frac{3s^2}{2} \ , \ s\frac{\sqrt{3}}{2} \ , \ p = r^2 + s^2 - rs) \ .$$

Eisenstein primes clearly span the ring of the complex integers having the general form $z = (r + i\sqrt{3}s)/2$, r and s integers.

One can use Eisenstein prime E_p to achieve the replacement of the $p_1^{-1/2}$-factor with $1/p_1$-factor in the partition functions Z_{p_1} the same effect for $p_1 \ mod \ 4 = 1$ and $p_1 \ mod \ 3 = 1$ with the net result that $i\sqrt{3}$ term appears. This trick does not work for $p_1 \ mod \ 4 = 3$ and $p_1 \ mod \ 3 = 2$. Note that the presence of *both* Gaussian and Eisenstein primes in the same factor Z_{p_1} is not allowed since in this case also the real part of Z_{p_1} would contain $\sqrt{3}$. This suggests that quite generally $p \ mod \ 4 = 1$ *resp.* $p \ mod \ 4 = 3 \wedge p \ mod \ 3 = 1$ parts of $\hat{\zeta}$ could correspond to Gaussian *resp.* Eisenstein primes.

For the factors Z_{p_1} satisfying $p_1 \ mod \ 4 = 3$ & $p_1 \ mod \ 3 = 2$ simultaneously, neither Gaussian nor Eisenstein primes can affect the rationalization of $p^{-n+1/2-iy}$ factor, and in this case more general algebraic extension of complex numbers is necessary as already found.

The algebraic extensions of rational numbers allow the notion of algebraic integer and prime quite generally [24]. In the general case however the decomposition of an algebraic integer into primes is not unique. In case of complex extensions of form $r + \sqrt{-d}s$ unique prime factorization is obtained only in nine cases corresponding to $d = 1, 2, 3, 7, 11, 19, 46, 67, 163$ [24]. $\sqrt{-d}$ corresponds to a root of unity only for $d = 1$ and $d = 3$, which perhaps makes Gaussian and Eisenstein primes special.

3.2 Tests for the $|\hat{\zeta}|^2 = |\zeta|^2$ hypothesis

The fact that the phases p_1^{iy} correspond to non-vanishing values of y, suggests that $|\hat{\zeta}|^2 = |\zeta|^2$ equality holds on the real axis only in the sense of a limiting procedure $y \rightarrow 0$. If the the values of y giving rise to allowed phases obey linear superposition (that is $k_1(p_1, y)$ defining the algebraic extension does not depend on y), the allowed values of y form a dense set of the real axis, since arbitrarily small differences $y_i - y_j$ are possible for the zeros of ζ. Hence the limiting procedure $y \rightarrow 0$ should be well-defined and give the expected answer if the basic hypothesis is correct.

3.2.1 What happens on the real axis?

The simplest test for the basic hypothesis is to look what happens on the real axis at the points $s = n$. Real ζ diverges at $s = 1$ and $s = 0$ and has trivial zeros are at the points $s = -2n$. The norm of $\hat{\zeta}$ is given by

$$|\hat{\zeta}(n)|_R = \prod_p \left[\prod_{p_1} |1 - p_1^{-n}|_p \right] . \tag{55}$$

For $n = 0$ a straightforward substitution to the formula implies that $|\hat{\zeta}(0)|$ vanishes. For $n > 0$ one has

$$|\hat{\zeta}(n)|_R = \prod_p \left[\prod_{p_1} |\frac{p_1^n - 1}{p_1^n}|_p \right] = \prod_p p^n \left[\prod_k \prod_{p_1^n \ mod \ p^k = 1} p^{-k} \right] . \tag{56}$$

Since the number of primes p_1 satisfying the condition $p_1^n \ mod \ p^k = 1$ is infinite, the norm vanishes for all values $n > 0$. For $s = -n < 0$ one has,

$$|\hat{\zeta}(n)|_R = \prod_p \left[\prod_{p_1} |1 - p_1^{n}|_p \right] . \tag{57}$$

and also this product vanishes always.

How to understand these results?

a) The results are consistent with the view that $|\zeta|_R$ on the real axis should be estimated by taking the limit $y \rightarrow 0$. Since the values of y in question involve necessarily differences of very large values of y, it is conceivable that the limiting procedure does not yield zero. That the limiting procedure can give zero for $Re[s] < 0$ could be partially due to the fact that for $Re[s] = -n < 0$ one has for the diagonal $p_1 = p$ contribution $|Z_p(-n+iy)|_p = 1$ whereas for $Re[s] = n > 0$ one has $|Z_p(n+iy)|_p > 1$ in general. Furthermore, for $Re[s] = -n$ only $p_1^n \ mod \ p^k = 1$ condition leads to the reduction of the p-adic norm of $Z_{p1 \neq p}$ whereas for $Re[s] = -2n$ also $p_1^n \ mod \ p^k = -1$ condition has the same effect.

b) One cannot exclude the possibility that only the proportionality $|\hat{\zeta}|^2 \propto |\zeta|^2$ holds true. For instance, in the super-conformal model predicting that the physical states of the model correspond to the zeros of ζ on the critical line, the Hermitian form defining the 'inner product' is proportional to the product of $sin(i\pi z)\Gamma(z)\zeta(z)$. This function vanishes for $Re[s] \notin \{0,1\}$ and the coefficient function of ζ is finite in the critical strip. For $s = 0$ this function however has the value $-1/2$ and for $s = 1$ the value is 1, whereas the naively evaluated value of $|\hat{\zeta}|$ vanishes identically at these points. Thus something else is necessarily involved.

c) It could also be that the product representation for the norm squared of $\hat{\zeta}$ as a product of its p-adic norms converges only in a restricted region. It would not be surprising if the negative values of y were excluded from the region of convergence for the representation of $|\hat{\zeta}|^2$ as a product of its p-adic norms. Concerning the proof of the Riemann hypothesis, the minimal requirement is that the region $[1/2 \leq Re[s] \leq 1 \ , \ y \neq 0]$ is included in the region of convergence.

One might think that $|\zeta|^2 = |\hat{\zeta}|^2$ hypothesis is testable simply by comparing the norm squared of the real zeta with the product of the p-adic norms of $|\hat{\zeta}|^2$. The problems are that the value for the product of p-adic norms is extremely sensitive to numerical errors since the p-adic norm of Pythagorean triangles phases fluctuates wildly as a function of the phase angle, and that one does not actually know what the values of p_1^{iy} actually are. One testable prediction, also following from the super-conformal model of the Riemann Zeta, is that the superpositions of the zeros are probably small values or minima of $|\zeta|_R$ on the lines $Re[s] = n/2$. More precisely, it is the function $G(1+iy_{12})$ which should have values smaller than one if the metric defined by G is Hermitian. One could also try to understand whether the the norm of $\hat{\zeta}$ allows a continuation to a continuous function of the complex argument identifiable as a modulus of an analytic function.

3.2.2 Can the imaginary part of $\hat{\zeta}$ vanish on the critical line?

Riemann Zeta is real on the critical line $Re[s] = 1/2$. A natural question is whether also $\hat{\zeta}$ has a vanishing imaginary part on this line. This is certainly not necessary since $\hat{\zeta}$ has values in the infinite-dimensional algebraic extension of rationals. It would be however highly desirable if this condition would hold true.

One cannot formulate the vanishing condition for the imaginary part in terms of the norm squared of any quantity defined by using the generalization of the adelic formula. The vanishing of the imaginary part of $\hat{\zeta}$ is however consistent with the Universality Principle. One can see this by expanding the factors $Z_{p_1} = 1/(1-p_1^{-1/2-iy})$ to a geometric series in powers of the irrational imaginary part of $p_1^{-1/2-iy}$. Each odd term in this series is proportional to $\sqrt{k(p_1,y)}$. One can combine the product of all these geometric series with the same value $k(p_1,y) = k$ to a sum of a rational part and an irrational part proportional to $\sqrt{k}$. If the irrational parts vanish separately for all allowed values of k, the imaginary part of $\hat{\zeta}$ indeed vanishes. This requires that the same value of $k(p_1,y) = k$ is associated with an infinite number of factors Z_{p_1}.

What is interesting is that the terms appearing in the sum over primes p_1 with the same value of k are proportional to $1/p_1^n$, $n \geq 1$: $n = 1$ terms are on the borderline at which the absolute convergence fails. If the number of primes p_1 with the same value of k is sufficiently small, also the sum over $n = 1$ terms with given k converges. The allowed values of k are given by $k = \sqrt{p_1 - m^2}$, $m \leq \sqrt{p_1}$ and the simplest hypothesis is that each value of k appears with same probability so that for a given prime p_1 the probability for a $k(p_1,y) = k$ is $P(k) \sim 1/\sqrt{p_1}$. This would suggest that the lowest terms in the sum defining the imaginary part behaves as $1/p_1^{3/2}$ so that convergence is indeed achieved. Note that convergence requirement does not support the special role of Gaussian or Eisenstein primes in the set of algebraic numbers appearing in the expansion of $\hat{\zeta}$.

The general algebraic properties of $\hat{\zeta}$ must be consistent with the vanishing of $Im[\zeta]$ on the critical line. The reality of ζ on the critical line follows from the symmetry with respect to the critical line reducing on the critical line to the condition $\zeta(s) = \zeta(1-s)$ implying the reality of $\zeta(s)\zeta(1-s)$. This condition makes sense also for $\hat{\zeta}$. In general case, one has

$$\hat{\zeta}(s)\hat{\zeta}(1-s) = \prod_{p_1} Z_{p_1}(x+iy)Z_{p_1}(1-x-iy) = \prod_{p_1} \frac{1}{\left[1 - p_1^{-x}p_1^{-iy} - p_1^{-1+x}p_1^{iy} + \frac{1}{p_1}\right]} \ .$$

Due to the presence of p^{-x} terms, the moduli squared for these factors are complex irrational numbers.

On the line $Re[s] = 1/2$, the product representation for this function reduces to the product of *real* factors

$$\frac{1}{Z_{p_1}(1/2+iy)Z_{p_1}(1/2-iy)} = 1 - p_1^{-1/2}(p_1^{iy} + p_1^{-iy}) + \frac{1}{p_1} \tag{58}$$

in the algebraic extension of rationals. Thus the reality and rationality of the function $\hat{\zeta}(s)\hat{\zeta}(1-s)$ on the critical line corresponds in a very transparent manner the reality of ζ on the critical line. Note also that the modulo p reality of the factors Z_{p_1} implied by the reduction of the p-adic norm can be regarded as the p-adic counterpart for the reality of ζ on the critical line.

3.2.3 What about non-algebraic zeros of ζ?

In principle real ζ could also have non-algebraic zeros. The following argument however demonstrates that they do not pose a problem. If Universality Principles holds true, and if the norm squared of $\hat{\zeta}$ defined as a product of its p-adic norms indeed equals to the norm squared of the real ζ in the set of of complex plane in which the factors $1/(1-p^{-s})$ are algebraic numbers, one obtains strict bounds for the norm of the real ζ excluding the zeros in the dense set inside the critical strip. The continuity of the real ζ in turn implies that it cannot vanish except on the critical line.

4 Riemann hypothesis and super-conformal invariance

Hilbert and Polya [20] conjectured a long time ago that the non-trivial zeroes of Riemann Zeta function could have spectral interpretation in terms of the eigenvalues of a suitable self-adjoint differential operator H such that the eigenvalues of this operator correspond to the imaginary parts of the nontrivial zeros $z = x + iy$ of ζ. One can however consider a variant of this hypothesis stating that the eigenvalue spectrum of a non-hermitian operator D^+ contains the non-trivial zeros of ζ. The eigen states in question are eigen states of an annihilation operator type operator D^+ and analogous to the so called coherent states encountered in quantum physics [25]. In particular, the eigenfunctions are in general non-orthogonal and this is a quintessential element of the the proposed strategy of proof.

In the following an explicit operator having as its eigenvalues the non-trivial zeros of ζ is constructed.

a) The construction relies crucially on the interpretation of the vanishing of ζ as an orthogonality condition in a hermitian metric which is is a priori more general than Hilbert space inner product.

b) Second basic element is the scaling invariance motivated by the belief that ζ is associated with a physical system which has super-conformal transformations [26] as its symmetries.

The core elements of the construction are following.

a) All complex numbers are candidates for the eigenvalues of D^+ (formal hermitian conjugate of D) and genuine eigenvalues are selected by the requirement that the condition $D^\dagger = D^+$ holds true in the set of the genuine eigenfunctions. This condition is equivalent with the hermiticity of the metric defined by a function proportional to ζ.

b) The eigenvalues turn out to consist of $z = 0$ and the non-trivial zeros of ζ and only the eigenfunctions corresponding to the zeros with $Re[s] = 1/2$ define a subspace possessing a hermitian metric. The vanishing of ζ tells that the 'physical' positive norm eigenfunctions (in general *not* orthogonal to each other), are orthogonal to the 'un-physical' negative norm eigenfunction associated with the eigenvalue $z = 0$.

The proof of the Riemann hypothesis by reductio ad absurdum results if one assumes that the space $\mathcal{V}$ spanned by the states corresponding to the zeros of ζ inside the critical strip has a hermitian induced metric. Riemann hypothesis follows also from the requirement that the induced metric in the spaces subspaces $\mathcal{V}_s$ of $\mathcal{V}$ spanned by the states Ψ_s and $\Psi_{1-\overline{s}}$ does not possess negative eigenvalues: this condition is equivalent with the positive definiteness of the metric in $\mathcal{V}$. Conformal invariance in the sense of gauge invariance allows only the states belonging to $\mathcal{V}$. Riemann hypothesis follows also from a restricted form of a dynamical conformal invariance in $\mathcal{V}$. This allows the reduction of the proof to a standard analytic argument used in Lie-group theory.

4.1 Modified form of the Hilbert-Polya conjecture

One can modify the Hilbert-Polya conjecture by assuming scaling invariance and giving up the hermiticity of the Hilbert-Polya operator. This means introduction of the non-hermitian operators D^+ and D which are hermitian conjugates of each other such that D^+ has the nontrivial zeros of ζ as its complex eigenvalues

$$D^+\Psi = z\Psi. \tag{59}$$

The counterparts of the so called coherent states [25] are in question and the eigenfunctions of D^+ are not expected to be orthogonal in general. The following construction is based on the idea that D^+ also allows the eigenvalue $z = 0$ and that the vanishing of ζ at z expresses the orthogonality of the states with eigenvalue $z = x + iy \neq 0$ and the state with eigenvalue $z = 0$ which turns out to have a negative norm.

The trial

$$\begin{aligned} &D = L_0 + V, \quad D^+ = -L_0 + V \\ &L_0 = t\tfrac{d}{dt}, \qquad V = \tfrac{dlog(F)}{d(log(t))} = t\tfrac{dF}{dt}\tfrac{1}{F} \end{aligned} \tag{60}$$

is motivated by the requirement of invariance with respect to scalings $t \rightarrow \lambda t$ and $F \rightarrow \lambda F$. The range of variation for the variable t consists of non-negative real numbers $t \geq 0$. The scaling invariance implying conformal invariance (Virasoro generator L_0 represents scaling which plays a fundamental role in the super-conformal theories [26]) is motivated by the belief that ζ codes for the physics of a quantum critical system having, not only super-symmetries [23], but also super-conformal transformations as its basic symmetries.

4.2 Formal solution of the eigenvalue equation for operator D^+

One can formally solve the eigenvalue equation

$$D^+\Psi_z = \left[-t\frac{d}{dt} + t\frac{dF}{dt}\frac{1}{F}\right]\Psi_z = z\Psi_z. \tag{61}$$

for D^+ by factoring the eigenfunction to a product:

$$\Psi_z = f_z F. \tag{62}$$

The substitution into the eigenvalue equation gives

$$L_0 f_z = t\frac{d}{dt} f_z = -z f_z \tag{63}$$

allowing as its solution the functions

$$f_z(t) = t^z. \tag{64}$$

These functions are nothing but eigenfunctions of the scaling operator L_0 of the super-conformal algebra analogous to the eigen states of a translation operator. A priori all complex numbers z are candidates for the eigenvalues of D^+ and one must select the genuine eigenvalues by applying the requirement $D^\dagger = D^+$ in the space spanned by the genuine eigenfunctions.

It must be emphasized that Ψ_z is *not* an eigenfunction of D. Indeed, one has

$$D\Psi_z = -D^+\Psi_z + 2V\Psi_z = z\Psi_z + 2V\Psi_z. \tag{65}$$

This is in accordance with the analogy with the coherent states which are eigen states of annihilation operator but not those of creation operator.

4.3 $D^+ = D^\dagger$ condition and hermitian form

The requirement that D^+ is indeed the hermitian conjugate of D implies that the hermitian form satisfies

$$\langle f|D^+g\rangle = \langle Df|g\rangle. \tag{66}$$

This condition implies

$$\langle \Psi_{z_1}|D^+\Psi_{z_2}\rangle = \langle D\Psi_{z_1}|\Psi_{z_2}\rangle. \tag{67}$$

The first (not quite correct) guess is that the hermitian form is defined as an integral of the product $\overline{\Psi}_{z_1}\Psi_{z_2}$ of the eigenfunctions of the operator D over the non-negative real axis using a suitable integration measure. The hermitian form can be defined by continuing the integrand from the non-negative real axis to the entire complex t-plane and noticing that it has a cut along the non-negative real axis. This suggests the definition of the hermitian form, not as a mere integral over the non-negative real axis, but as a contour integral along curve C defined so that it encloses the non-negative real axis, that is C

a) traverses the non-negative real axis along the line $Im[t] = 0_-$ from $t = \infty + i0_-$ to $t = 0_+ + i0_-$,

b) encircles the origin around a small circle from $t = 0_+ + i0_-$ to $t = 0_+ + i0_+$,

c) traverses the non-negative real axis along the line $Im[t] = 0_+$ from $t = 0_+ + i0_+$ to $t = \infty + i0_+$.

Here $0_\pm$ signifies taking the limit $x = \pm\epsilon$, $\epsilon > 0$, $\epsilon \to 0$.

C is the correct choice if the integrand defining the inner product approaches zero sufficiently fast at the limit $Re[t] \to \infty$. Otherwise one must assume that the integration contour continues along the circle S_R of radius $R \to \infty$ back to $t = \infty + i0_-$ to form a closed contour. It however turns out that this is not necessary. One can deform the integration contour rather freely: the only constraint is that the deformed integration contour does not cross over any cut or pole associated with the analytic continuation of the integrand from the non-negative real axis to the entire complex plane.

Scaling invariance dictates the form of the integration measure appearing in the hermitian form uniquely to be dt/t. The hermitian form thus obtained also makes possible to satisfy the crucial $D^+ = D^\dagger$ condition. The hermitian form is thus defined as

$$\langle \Psi_{z_1} | \Psi_{z_2} \rangle = -\frac{K(z_{12})}{2\pi i} \int_C \overline{\Psi_{z_1}} \Psi_{z_2} \frac{dt}{t}. \tag{68}$$

$K(z_{12})$ is real from the hermiticity requirement and the behavior as a function of $z_{12} = z_1 + \overline{z}_2$ by the requirement that the resulting Hermitian form defines a positive definite inner product. The value of $K(1)$ can can be fixed by requiring that the states corresponding to the zeros of ζ at the critical line have unit norm: with this choice the vacuum state corresponding to $z = 0$ has negative norm. Physical intuition suggests that $K(z_{12})$ is responsible for the Gaussian overlaps of the coherent states and this suggests the behavior

$$K(z_{12}) = exp(-\alpha |z_{12}|^2), \tag{69}$$

for which overlaps between states at critical line are proportional to $exp(-\alpha(y_1-y_2)^2)$ so that for $\alpha > 0$ Schwartz inequalities are certainly satisfied for large values of $|y_{12}|$. Small values of y_{12} are dangerous in this respect but since the matrix elements of the metric decrease for small values of y_{12} even for $K(z_{12}) = 1$, it is possible to satisfy Schwartz inequalities for sufficiently large value of α. It must be emphasized that the detailed behavior

of K is *not* crucial for the arguments relating to Riemann hypothesis.

The possibility to deform the shape of C in wide limits realizes conformal invariance stating that the change of the shape of the integration contour induced by a conformal transformation, which is nonsingular inside the integration contour, leaves the value of the contour integral of an analytic function unchanged. This scaling invariant hermitian form is indeed a correct guess. By applying partial integration one can write

$$\langle \Psi_{z_1} | D^+ \Psi_{z_2} \rangle = \langle D\Psi_{z_1} | \Psi_{z_2} \rangle - \frac{K(z_{12})}{2\pi i} \int_C dt \frac{d}{dt} \left[\overline{\Psi}_{z_1}(t) \Psi_{z_2}(t) \right] . \tag{70}$$

The integral of a total differential comes from the operator $L_0 = td/dt$ and must vanish. For a non-closed integration contour C the boundary terms from the partial integration could spoil the $D^+ = D^\dagger$ condition unless the eigenfunctions vanish at the end points of the integration contour ($t = \infty + i0_\pm$).

The explicit expression of the hermitian form is given by

$$\begin{aligned} \langle \Psi_{z_1} | \Psi_{z_2} \rangle &= -\frac{K(z_{12})}{2\pi i} \int_C \frac{dt}{t} F^2(t) t^{z_{12}} , \\ z_{12} &= \overline{z}_1 + z_2 . \end{aligned} \tag{71}$$

It must be emphasized that it is $\overline{\Psi}_{z_1} \Psi_{z_2}$ rather than eigenfunctions which is continued from the non-negative real axis to the complex t-plane: therefore one indeed obtains an analytic function as a result.

An essential role in the argument claimed to prove the Riemann hypothesis is played by the crossing symmetry

$$\langle \Psi_{z_1} | \Psi_{z_2} \rangle = \langle \Psi_0 | \Psi_{\overline{z}_1 + z_2} \rangle \tag{72}$$

of the hermitian form. This symmetry is analogous to the crossing symmetry of particle physics stating that the S-matrix is symmetric with respect to the replacement of the particles in the initial state with their antiparticles in the final state or vice versa [25].

The hermiticity of the hermitian form implies

$$\langle \Psi_{z_1} | \Psi_{z_2} \rangle = \overline{\langle \Psi_{z_2} | \Psi_{z_1} \rangle}. \tag{73}$$

This condition, which is *not* trivially satisfied, in fact determines the eigenvalue spectrum.

4.4 How to choose the function F?

The remaining task is to choose the function F in such a manner that the orthogonality conditions for the solutions Ψ_0 and Ψ_z reduce to the condition that ζ or some function proportional to ζ vanishes at the point $-z$. The definition of ζ based on analytical continuation performed by Riemann suggests how to proceed. Recall that the expression of ζ converging in the region $Re[s] > 1$ following from the basic definition of ζ and elementary properties of Γ function [27] reads as

$$\Gamma(s)\zeta(s) = \int_0^\infty \frac{dt}{t} \frac{exp(-t)}{[1 - exp(-t)]} t^s. \tag{74}$$

One can analytically continue this expression to a function defined in the entire complex plane by noticing that the integrand is discontinuous along the cut extending from $t = 0$ to $t = \infty$. Following Riemann it is however more convenient to consider the discontinuity for a function obtained by multiplying the integrand with the factor

$$(-1)^s \equiv exp(-i\pi s).$$

The discontinuity $Disc(f) \equiv f(t) - f(texp(i2\pi))$ of the resulting function is given by

$$Disc\left[\frac{exp(-t)}{[1 - exp(-t)]}(-t)^{s-1}\right] = -2isin(\pi s)\frac{exp(-t)}{[1 - exp(-t)]}t^{s-1}. \tag{75}$$

The discontinuity vanishes at the limit $t \to 0$ for $Re[s] > 1$. Hence one can define ζ by modifying the integration contour from the non-negative real axis to an integration contour C enclosing non-negative real axis defined in the previous section.

This amounts to writing the analytical continuation of $\zeta(s)$ in the form

$$-2i\Gamma(s)\zeta(s)sin(\pi s) = \int_C \frac{dt}{t}\frac{exp(-t)}{[1-exp(-t)]}(-t)^{s-1}. \tag{76}$$

This expression equals to $\zeta(s)$ for $Re[s] > 1$ and defines $\zeta(s)$ in the entire complex plane since the integral around the origin eliminates the singularity.

The crucial observation is that the integrand on the righthand side of Eq. 76 has precisely the same general form as that appearing in the hermitian form defined in Eq. 71 defined using the same integration contour C. The integration measure is dt/t, the factor t^s is of the same form as the factor $t^{\overline{z}_1+z_2}$ appearing in the hermitian form, and the function $F^2(t)$ is given by

$$F^2(t) = \frac{exp(-t)}{1-exp(-t)}.$$

Therefore one can make the identification

$$F(t) = \left[\frac{exp(-t)}{1-exp(-t)}\right]^{1/2}. \tag{77}$$

Note that the argument of the square root is non-negative on the non-negative real axis and that $F(t)$ decays exponentially on the non-negative real axis and has $1/\sqrt{t}$ type singularity at origin. From this it follows that the eigenfunctions $\Psi_z(t)$ approach zero exponentially at the limit $Re[t] \to \infty$ so that one can use the non-closed integration contour C.

With this assumption, the hermitian form reduces to the expression

$$\begin{aligned} \langle\Psi_{z_1}|\Psi_{z_2}\rangle &= -\frac{K(z_{12})}{2\pi i}\int_C \frac{dt}{t}\frac{exp(-t)}{[1-exp(-t]}(-t)^{z_{12}} \\ &= \frac{K(z_{12})}{\pi}sin(\pi z_{12})\Gamma(z_{12})\zeta(z_{12}). \end{aligned} \tag{78}$$

Recall that the definition $z_{12} = \overline{z}_1 + z_2$ is adopted. Thus the orthogonality of the eigenfunctions is equivalent to the vanishing of $\zeta(z_{12})$ if $K(z_{12})$ is positive definite.

4.5 Study of the hermiticity condition

In order to derive information about the spectrum one must explicitly study what the statement that $D^\dagger$ is hermitian conjugate of D means. The defining equation is just the generalization of the equation

$$A^\dagger_{mn} = \overline{A}_{nm}. \tag{79}$$

defining the notion of hermiticity for matrices. Now indices m and n correspond to the eigenfunctions Ψ_{z_i}, and one obtains

$$\langle \Psi_{z_1} | D^+ \Psi_{z_2} \rangle = z_2 \langle \Psi_{z_1} | \Psi_{z_2} \rangle = \overline{\langle \Psi_{z_2} | D \Psi_{z_1} \rangle} = \overline{\langle D^+ \Psi_{z_2} | \Psi_{z_1} \rangle} = z_2 \overline{\langle \Psi_{z_2} | \Psi_{z_1} \rangle}.$$

Thus one has

$$\begin{aligned} G(z_{12}) &= \overline{G(z_{21})} = \overline{G(\overline{z}_{12})} \\ G(z_{12}) &\equiv \langle \Psi_{z_1} | \Psi_{z_2} \rangle . \end{aligned} \tag{80}$$

The condition states that the hermitian form defined by the contour integral is indeed hermitian. This is *not* trivially true. Hermiticity condition obviously determines the spectrum of the eigenvalues of D^+.

To see the implications of the hermiticity condition, one must study the behavior of the function $G(z_{12})$ under complex conjugation of both the argument and the value of the function itself. To achieve this one must write the integral

$$G(z_{12}) = -\frac{K(z_{12})}{2\pi i} \int_C \frac{dt}{t} \frac{exp(-t)}{[1 - exp(-t)]} (-t)^{z_{12}}$$

in a form from which one can easily deduce the behavior of this function under complex conjugation. To achieve this, one must perform the change $t \rightarrow u = log(exp(-i\pi)t)$ of the integration variable giving

$$G(z_{12}) = -\frac{K(z_{12})}{2\pi i} \int_D du \frac{exp(-exp(u))}{[1 - exp(-(exp(u)))]} exp(z_{12}u). \tag{81}$$

Here D denotes the image of the integration contour C under $t \rightarrow u = log(-t)$. D is a fork-like contour which

a) traverses the line $Im[u] = i\pi$ from $u = \infty + i\pi$ to $u = -\infty + i\pi$,
b) continues from $-\infty + i\pi$ to $-\infty - i\pi$ along the imaginary u-axis (it is easy to see that the contribution from this part of the contour vanishes),
c) traverses the real u-axis from $u = -\infty - i\pi$ to $u = \infty - i\pi$,

The integrand differs on the line $Im[u] = \pm i\pi$ from that on the line $Im[u] = 0$ by the factor $exp(\mp i\pi z_{12})$ so that one can write $G(z_{12})$ as integral over real u-axis

$$G(z_{12}) = -\frac{K(z_{12})}{\pi} sin(\pi z_{12}) \int_{-\infty}^{\infty} du \frac{exp(-exp(u))}{[1 - exp(-(exp(u)))]} exp(z_{12}u). \tag{82}$$

From this form the effect of the transformation $G(z) \rightarrow \overline{G(\overline{z})}$ can be deduced. Since the integral is along the real u-axis, complex conjugation amounts only to the replacement $z_{21} \rightarrow z_{12}$, and one has

$$\begin{aligned} \overline{G(\overline{z}_{12})} &= -\frac{\overline{K(z_{21})}}{\pi} \times \overline{sin(\pi z_{21})} \int_{-\infty}^{\infty} du \frac{exp(-exp(u))}{[1 - exp(-(exp(u)))]} exp(z_{12}u) \\ &= \frac{\overline{K(z_{21})}}{K(z_{12})} \times \frac{\overline{sin(\pi z_{21})}}{sin(\pi z_{12})} G(z_{12}). \end{aligned} \tag{83}$$

Thus the hermiticity condition reduces to the condition

$$G(z_{12}) = \frac{\overline{K(z_{21})}}{K(z_{12})} \times \frac{\overline{sin(\pi z_{21})}}{sin(\pi z_{12})} \times G(z_{12}). \tag{84}$$

The reality of $K(z_{12})$ guarantees that the diagonal matrix elements of the metric are real.

For non-diagonal matrix elements there are two manners to satisfy the hermiticity condition.

a) The condition

$$G(z_{12}) = 0 \tag{85}$$

is the only manner to satisfy the hermiticity condition for $x_1 + x_2 \neq n$, $y_1 - y_2 \neq 0$. This implies the vanishing of ζ:

$$\zeta(z_{12}) = 0 \text{ for } 0 < x_1 + x_2 < 1. \quad (86)$$

In particular, this condition must be true for $z_1 = 0$ and $z_2 = 1/2 + iy$. Hence the physical states with the eigenvalue $z = 1/2 + iy$ must correspond to the zeros of ζ.

b) For the non-diagonal matrix elements of the metric the condition

$$exp(i\pi(x_1 + x_2)) = \pm 1 \quad (87)$$

guarantees the reality of $sin(\pi z_{12})$ factors. This requires

$$x_1 + x_2 = n. \quad (88)$$

The highly non-trivial implication is that the the vacuum state Ψ_0 and the zeros of ζ at the critical line span a space having a hermitian. Note that for $x_1 = x_2 = n/2$, $n \neq 1$, the diagonal matrix elements of the metric vanish.

c) The metric is positive definite only if the function $K(z_{12})$ decays sufficiently fast: this is due to the exponential increase of the moduli of the matrix elements $G(1/2+iy_1, 1/2+iy_2)$ for $K(z_{12}) = 1$ and for large values of $|y_1 - y_2|$ (basically due to the $sinh\,[\pi\,(y_1 - y_2)]$-factor in the metric) implying the failure of the Schwartz inequality for $|y_1 - y_2| \rightarrow \infty$. Unitarity, guaranteing probability interpretation in quantum theory, thus requires that the parameter α characterizing the Gaussian decay of $K(z_{12}) = exp(-\alpha |z_{12}|^2)$ is above some minimum value.

4.6 Various assumptions implying Riemann hypothesis

As found, the general strategy for proving the Riemann hypothesis, originally inspired by super-conformal invariance, leads to the construction of a set of eigen states for an operator D^+, which is effectively an annihilation operator acting in the space of complex-valued functions defined on the real half-line. Physically the states are analogous to coherent states and are not orthogonal to each other. The quantization of the eigenvalues for the operator D^+ follows from the requirement that the metric, which is defined by the integral defining the analytical continuation of ζ, and thus proportional to ζ ($\langle s_1, s_2 \rangle \propto \zeta(\overline{s}_1 + s_2)$), is hermitian in the space of the physical states.

The nontrivial zeros of ζ are known to belong to the critical strip defined by $0 < Re[s] < 1$. Indeed, the theorem of Hadamard and de la Vallee Poussin

[28] states the non-vanishing of ζ on the line $Re[s] = 1$. If s is a zero of ζ inside the critical strip, then also $1-\overline{s}$ as well as $\overline{s}$ and $1-s$ are zeros. If Hilbert space inner product property is not required so that the eigenvalues of the metric tensor can be also negative in this subspace. There could be also un-physical zeros of ζ outside the critical line $Re[s] = 1/2$ but inside the critical strip $0 < Re[s] < 1$. The problem is to find whether the zeros outside the critical line are excluded, not only by the hermiticity but also by the positive definiteness of the metric necessary for the physical interpretation, and perhaps also by conformal invariance posed in some sense as a dynamical symmetry. This turns out to be the case.

Before continuing it is convenient to introduce some notations. Denote by $\mathcal{V}$ the subspace spanned by Ψ_s corresponding to the zeros s of ζ inside the critical strip, by $\mathcal{V}_{crit}$ the subspace corresponding to the zeros of ζ at the critical strip, and by $\mathcal{V}_s$ the space spanned by the states Ψ_s and $\Psi_{1-\overline{s}}$. The basic idea behind the following proposals is that the basic objects of study are the spaces $\mathcal{V}$, $\mathcal{V}_{crit}$ and $\mathcal{V}_s$.

4.6.1 How to restrict the metric to $\mathcal{V}$?

One should somehow restrict the metric defined in the space spanned by the states Ψ_s labelled by a continuous complex eigenvalue s to the space $\mathcal{V}$ inside the critical strip spanned by a basis labelled by discrete eigenvalues. Very naively, one could try to do this by simply putting all other components of the metric to zero so that the states outside $\mathcal{V}$ correspond to gauge degrees of freedom. This is consistent with the interpretation of $\mathcal{V}$ as a coset space formed by identifying states which differ from each other by the addition of a superposition of states which do not correspond to zeros of ζ.

An more elegant manner to realize the restriction of the metric to $\mathcal{V}$ is to Fourier expand states in the basis labelled by a complex number s and define the metric in $\mathcal{V}$ using double Fourier integral over the complex plane and Dirac delta function restricting the labels of both states to the set of zeros inside the critical strip:

$$
\begin{aligned}
\langle \Psi^{1)} | \Psi^{2)} \rangle &= \int d\mu(s_1) \int d\mu(s_2) \overline{\Psi}^{1)}_{s_1} \Psi^{2)}_{s_2} G(s_2+\overline{s}_1)\delta(\zeta(s_1))\delta(\zeta(s_2)) \\
&= \sum_{\zeta(s_1)=0,\zeta(s_2)=0} \overline{\Psi}^{1)}_{s_1} \Psi^{2)}_{s_2} G(s_2+\overline{s}_1) \frac{1}{\sqrt{det(s_2)det(\overline{s}_1)}} , \\
d\mu(s) &= ds d\overline{s} , \qquad det(s) = \frac{\partial(Re\left[\zeta(s)\right], Im\left[\zeta(s)\right])}{\partial(Re\left[s\right], Im\left[s\right])} . \qquad (89)
\end{aligned}
$$

Here the integrations are over the critical strip. $det(s)$ is the Jacobian for the map $s \rightarrow \zeta(s)$ at s. The appearance of the determinants might be crucial for the absence of negative norm states. The result means that the metric $G_{\mathcal{V}}$ in $\mathcal{V}$ effectively reduces to a product

$$\begin{aligned} G_{\mathcal{V}} &= \overline{D}GD, \\ D(s_i, s_j) &= D(s_i)\delta(s_i, s_j), \\ \overline{D}(s_i, s_j) &= D(\overline{s_i})\delta(s_i, s_j) \\ D(s) &= \frac{1}{\sqrt{det(s)}}. \end{aligned} \tag{90}$$

In the sequel the metric G will be called reduced metric whereas $G_{\mathcal{V}}$ will be called the full metric. In fact, the symmetry $D(s) = D(\overline{s})$ holds true by the basic symmetries of ζ so that one has $D = \overline{D}$ and $G_{\mathcal{V}} = DGD$. This means that Schwartz inequalities for the eigen states of D^+ are not affected in the replacement of $G_{\mathcal{V}}$ with G. The two metrics can be in fact transformed to each other by a mere scaling of the eigen states and are in this sense equivalent.

4.6.2 Riemann hypothesis from the hermicity of the metric in $\mathcal{V}$

The mere requirement that the metric is hermitian in $\mathcal{V}$ implies the Riemann hypothesis. This can be seen in the simplest manner as follows. Besides the zeros at the critical line $Re[s] = 1/2$ also the symmetrically related zeros inside critical strip have positive norm squared but they do not have hermitian inner products with the states at the critical line unless one assumes that the inner product vanishes. The assumption that the inner products between the states at critical line and outside it vanish, implies additional zeros of ζ and, by repeating the argument again and again, one can fill the entire critical interval $(0, 1)$ with the zeros of ζ so that a reductio ad absurdum proof for the Riemann hypothesis results. Thus the metric gives for the states corresponding to the zeros of the Riemann Zeta at the critical line a special status as what might be called physical states.

It should be noticed that the states in $\mathcal{V}_s$ and $\mathcal{V}_{\overline{s}}$ have non-hermitian inner products for $Re[s] \neq 1/2$ unless these inner products vanish: for $Re[s] > 1/2$ this however implies that ζ has a zero for $Re[s] > 1$.

4.6.3 Riemann hypothesis from the requirement that the metric in $\mathcal{V}$ is positive definite

With a suitable choice of $K(z_{12})$ the metric is positive definite between states having $y_1 \neq y_2$. For s and $1-\overline{s}$ one has $y_1 = y_2$ implying $K(z_{12}) = 1$ in $\mathcal{V}_s$. Thus the positive definiteness of the metric in $\mathcal{V}$ reduces to that for the induced metric in the spaces $\mathcal{V}_s$. This requirement implies also Riemann hypothesis as following argument shows.

The explicit expression for the norm of a $Re[s] = 1/2$ state with respect to the full metric $G^{ind}_{\mathcal{V}}$ reads as

$$\begin{aligned} G^{ind}_{\mathcal{V}}(1/2+iy_n, 1/2+iy_n) &= D^2(1/2+iy)G^{ind}(1/2+iy_n, 1/2+iy_n), \\ G^{ind}(1/2+iy_n, 1/2+iy_n) &= -\frac{K(z_{12})}{\pi} sin(\pi)\Gamma(1)\zeta(1). \end{aligned} \tag{91}$$

Here G^{ind} is the metric in $\mathcal{V}_s$ induced from the reduced metric G. This expression involves formally a product of vanishing and infinite factors and the value of expression must be defined as a limit by taking in $Im[z_{12}]$ to zero. The requirement that the norm squared defined by G^{ind} equals to one fixes the value of $K(1)$:

$$K(1) = -\frac{\pi}{sin(\pi)\zeta(1)} = 1. \tag{92}$$

The components G^{ind} in $\mathcal{V}_s$ are given by

$$\begin{aligned} G^{ind}(s,s) &= -\frac{sin(2\pi Re[s])\Gamma(2Re[s])\zeta(2Re[s])}{\pi}, \\ G^{ind}(1-\overline{s}, 1-\overline{s}) &= -\frac{sin(2\pi(1-Re[s]))\Gamma(2-2Re[s])\zeta(2(1-[Re[s]))}{\pi}, \\ G^{ind}(s, 1-\overline{s}) &= G^{ind}(1-\overline{s}, s) = 1. \end{aligned} \tag{93}$$

The determinant of the metric $G^{ind}_{\mathcal{V}}$ induced from the full metric reduces to the product

$$Det(G^{ind}_{\mathcal{V}}) = D^2(s))D^2(1-\overline{s}) \times Det(G^{ind}). \tag{94}$$

Since the first factor is positive definite, it suffices to study the determinant of G^{ind}. At the limit $Re[s] = 1/2$ G^{ind} formally reduces to

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} .$$

This reflects the fact that the states Ψ_s and $\Psi_{1-\overline{s}}$ are identical. The actual metric is of course positive definite. For $Re[s] = 0$ the G^{ind} is of the form

$$\begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} .$$

The determinant of G^{ind} is negative so that the eigenvalues of both the full metric and reduced metric are of opposite sign. The eigenvalues for G^{ind} are given by $(-1 \pm \sqrt{5})/2$.

The determinant of G^{ind} in $\mathcal{V}_s$ as a function of $Re[s]$ is symmetric with respect to $Re[s] = 1/2$, equals to -1 at the end points $Re[s] = 0$ and $Re[s] = 1$, and vanishes at $Re[s] = 1/2$. Numerical calculation shows that the sign of the determinant of G^{ind} inside the interval $(0, 1)$ is negative for $Re[s] \neq 1/2$. Thus the diagonalized form of the induced metric has the signature $(1, -1)$ except at the limit $Re[s] = 1/2$, when the signature formally reduces to $(1, 0)$. Thus Riemann hypothesis follows if one can show that the metric induced to $\mathcal{V}_s$ does not allow physical states with a negative norm squared. This requirement is physically very natural. In fact, when the factor $K(z_{12})$ represents sufficiently rapidly vanishing Gaussian, this guarantees the metric to $\mathcal{V}_{crit}$ has only non-negative eigenvalues. Hence the positive-definiteness of the metric, natural if there is real quantum system behind the model, implies Riemann hypothesis.

4.6.4 Riemann hypothesis and conformal invariance

The basic strategy for proving Riemann hypothesis has been based on the attempt to reduce Riemann hypothesis to invariance under conformal algebra or some subalgebra of the conformal algebra in $\mathcal{V}$ or $\mathcal{V}_s$. That this kind of algebra should act as a gauge symmetry associated with ζ is very natural idea since conformal invariance is in a well-defined sense the basic symmetry group of complex analysis.

Consider now one particular strategy based on conformal invariance in the space of the eigen states of D^+.

1. Realization of conformal algebra as a spectrum generating algebra

The conformal generators are realized as operators

$$L_z = t^z D^+ \tag{95}$$

act in the eigen space of D^+ and obey the standard conformal algebra without central extension [26]. D^+ itself corresponds to the conformal generator L_0 acting as a scaling. Conformal generators obviously act as dynamical symmetries transforming eigen states of D^+ to each other. What is new is that now conformal weights z have all possible complex values unlike in the standard case in which only integer values are possible. The vacuum state Ψ_0 having negative norm squared is annihilated by the conformal algebra so that the states orthogonal to it (non-trivial zeros of ζ inside the critical strip) form naturally another subspace which should be conformally invariant in some sense. Conformal algebra could act as gauge algebra and some subalgebra of the conformal algebra could act as a dynamical symmetry.

2. Realization of conformal algebra as gauge symmetries

The definition of the metric in $\mathcal{V}$ involves in an essential manner the mapping $s \rightarrow \zeta(s)$. This suggests that one should define the gauge action of the conformal algebra as

$$\begin{aligned} \Psi_s &\rightarrow \Psi_{\zeta(s)} \rightarrow L_z \Psi_{\zeta(s)} = \zeta_s \Psi_{\zeta(s)+z} \\ &\rightarrow \zeta_s \Psi_{\zeta^{-1}(\zeta(s)+z)} . \end{aligned} \tag{96}$$

Clearly, the action involves a map of the conformal weight s to $\zeta(s)$, the action of the conformal algebra to $\zeta(s)$, and the mapping of the transformed conformal weight $z+\zeta(s)$ back to the complex plane by the inverse of ζ. The inverse image is in general non-unique but in case of $\mathcal{V}$ this does not matter since the action annihilates automatically all states in $\mathcal{V}$. Thus conformal algebra indeed acts as a gauge symmetry. This symmetry does not however force Riemann hypothesis.

3. Realization of conformal algebra as dynamical symmetries

One can also study the action of the conformal algebra or its suitable sub-algebra in $\mathcal{V}_s$ as a dynamical (as opposed to gauge) symmetry realized as

$$\Psi_s \rightarrow L_z \Psi_s = s \Psi_{s+z} . \tag{97}$$

The states Ψ_s and $\Psi_{1-\overline{s}}$ in $\mathcal{V}_s$ have non-vanishing norms and are obtained from each other by the conformal generators $L_{1-2Re[s]}$ and $L_{2Re[s]-1}$. For $Re[s] \neq 1/2$ the generators $L_{1-2Re[s]}$, $L_{2Re[s]-1}$, and L_0 generate $SL(2,R)$ algebra which is non-compact and generates infinite number of states from the states of $\mathcal{V}_s$. At the critical line this algebra reduces to the abelian algebra spanned by L_0. The requirement that the algebra naturally associated with $\mathcal{V}_s$ is a dynamical symmetry and thus generates only zeros of ζ leads to the conclusion that all points $s+n(1-2Re[s])$, n integer, must be zeros of ζ. Clearly, $Re[s]=1/2$ is the only possibility so that Riemann hypothesis follows. In this case the dynamical symmetry indeed reduces to a gauge symmetry.

There is clearly a connection with the argument based on the requirement that the induced metric in $\mathcal{V}_s$ does not possess negative eigenvalues. Since $SL(2,R)$ algebra acts as the isometries of the induced metric for the zeros having $Re[s]\neq 1/2$, the signature of the induced metric must be $(1,-1)$.

4. Riemann hypothesis from the requirement that infinitesimal isometries exponentiate

One could even try to prove that the entire subalgebra of the conformal algebra spanned by the generators with conformal weights $n(1-2Re[s])$ acts as a symmetry generating new zeros of ζ so that corresponding states are annihilated by gauge conformal algebra. If this holds, $Re[s]=1/2$ is the only possibility so that Riemann hypothesis follows. In this case the dynamical conformal symmetry indeed reduces to a gauge symmetry.

Since $L_{1-2Re[s]}$ acts as an infinitesimal isometry leaving the matrix element $\langle\Psi_0|\Psi_s\rangle=0$ invariant, one can in spirit of Lie group theory argue that also the exponentiated transformations $exp(tL_{1-2Re[s]})$ have the same property for all values of t. The exponential action leaves Ψ_0 invariant and generates from Ψ_s a superposition of states with conformal weights $s+n(1-2Re[s])$, which all must be orthogonal to Ψ_0 since t is arbitrary. Since all zeros are inside the critical strip, $Re[s]=1/2$ is the only possibility.

A more explicit formulation of this idea is based on a first order differential equation for the integral representation of ζ. One can write the matrix element of the metric using the analytical continuation of $\zeta(s)$:

$$\begin{aligned} G(s) &= -2i\Gamma(s)\zeta(s)sin(\pi s)=H(s,a)_{|a=0} , \\ H(s,a) &= \int_C \frac{dt}{t}\frac{exp(-t+a(-t)^{1-2x})}{[1-exp(-t)]}(-t)^{x+iy-1} . \end{aligned} \tag{98}$$

If $s=x+iy$ is zero of ζ then also $1-x+iy$ is zero of ζ and its is trivial to see that this means the both $H(x+iy,a)$ and its first derivative vanishes at $a=0$:

$$\begin{aligned} H(s,a)_{|a=0} = 0 , \\ \frac{d}{da}H(s,a)_{|a=0} = 0 . \end{aligned} \tag{99}$$

Suppose that $H(s,a)$ satisfies a differential equation of form

$$\frac{d}{da}H(x+iy,a) = I(x, H(x+iy,a)), \tag{100}$$

where $I(x,H)$ is some function having no explicit dependence on a so that the differential equation defines an autonomous flow. If the initial conditions of Eq. 99 are satisfied, this differential equation implies that all derivatives of H vanish which in turn, as it is easy to see, implies that the points $s + m(1-2x)$ are zeros of ζ. This leaves only the possibility $x = 1/2$ so that Riemann hypothesis is proven. If I is function of also a, that is $I = I(a,x,H)$, this argument breaks down.

The following argument shows that the system is autonomous. One can solve a as function $a = a(x,H)$ from the Taylor series of H with respect to a by using implicit function theorem, substitute this series to the Taylor series of dH/da with respect to a, and by re-organizing the summation obtain a Taylor series with respect to H with coefficients which depend only on x so that one has $I = I(x,H)$.

5. Conclusions

To sum up, Riemann hypothesis follows from the requirement that the states in $\mathcal{V}$ can be assigned with a conformally invariant physical quantum system. This condition reduces to three mutually equivalent conditions: the metric induced to $\mathcal{V}$ is hermitian; positive definite; allows conformal symmetries as isometries. The hermiticity and positive definiteness properties reduce to the requirement that the dynamical conformal algebra naturally spanned by the states in $\mathcal{V}_s$ reduces to the abelian algebra defined by $L_0 = D^+$. If the infinitesimal isometries for the matrix elements $\langle\Psi_0|\Psi_s\rangle = 0$ generated by $L_{1-2Re[s]}$ can be exponentiated to isometries as Lie group theory based argument strongly suggests, then Riemann hypothesis follows.

4.7 Does the Hermitian form define inner product?

Before considering the question whether the Hermitian form defined by G or $G_{\mathcal{V}}$ defines a positive definite Hilbert space inner product, a couple of comments concerning the general properties of the Hermitian form G are in order.

a) The Hermitian form is proportional to the factor

$$sin(i\pi(y_2 - y_1)) \ ,$$

which vanishes for $y_1 = y_2$. For $y_1 = y_2$ and $x_1 + x_2 = 1$ ($x_1 + x_2 = 0$) the diverging factor $\zeta(1)$ ($\zeta(0)$) compensates the vanishing of this factor. Therefore the norms of the eigenfunctions Ψ_z with $z = 1/2 + iy$ must be calculated explicitly from the defining integral. Since the contribution from the cut vanishes in this case, one obtains only an integral along a small circle around the origin. This gives the result

$$\langle \Psi_{z_1} | \Psi_{z_1} \rangle = K \;\text{ for }\; z_1 = \frac{1}{2} + iy \;\; , \;\; \langle \Psi_0 | \Psi_0 \rangle = -K \;\; . \tag{101}$$

Thus the norms of the eigenfunctions are finite. For $K = 1$ the norms of $z = 1/2 + iy$ eigenfunctions are equal to one. Ψ_0 has however negative norm -1 so that the Hermitian form in question is not a genuine inner product in the space containing Ψ_0.

b) For $x_1 = x_2 = 1/2$ and $y_1 \neq y_2$ the factor is non-vanishing and one has

$$\langle \Psi_{z_1} | \Psi_{z_2} \rangle = -\frac{1}{\pi i}\zeta(1 + i(y_2 - y_1))\Gamma(1 + i(y_2 - y_1))sinh(\pi(y_2 - y_1)) \;\; . \tag{102}$$

The nontrivial zeros of ζ are known to belong to the critical strip defined by $0 < Re[s] < 1$. Indeed, the theorem of Hadamard and de la Vallee Poussin [28] states the non-vanishing of ζ on the line $Re[s] = 1$. Since the non-trivial zeros of ζ are located symmetrically with respect to the line $Re[s] = 1/2$, this implies that the line $Re[s] = 0$ cannot contain zeros of ζ. This result implies that the states $\Psi_{z=1/2+y}$ are non-orthogonal unless $\Gamma(1 + i(y_2 - y_1))$ vanishes for some pair of eigenfunctions.

It is not at all obvious that the Hermitian form in question defines an inner product in the space spanned by the states Ψ_z, $z = 1/2 + iy$ having real and positive norm. Besides Hermiticity, a necessary condition for this is that Schwartz inequality

$$|\langle \Psi_{z_1} | \Psi_{z_2} \rangle| \leq |\Psi_{z_1}||\Psi_{z_2}|$$

holds true. In case of eigen states of D^+ this condition is not affected by the determinant factors and one can apply it to the metric G. This gives

$$\frac{1}{\pi}|\zeta(1 + iy_{12})| \times |\Gamma(1 + iy_{12})| \times |sin(i\pi y_{12})| \leq 1 \;\; , \tag{103}$$

where the shorthand notation $y_{12} = y_2 - y_1$ has been used.

Numerical computation suggests that $\zeta(1+iy_{12})$ varies in a finite range of values for large values of y_{12} and that $\Gamma(1+iy)$ behaves essentially as $exp(-\pi y/2)$ asymptotically so that the left hand side increases faster than $exp(\pi y_{12}/2)$ so that Schwartz inequality fails for the eigen states. It took a considerable time do realize that the solution to this difficulty is trivial: the only thing that is needed is to multiply the metric with the factor $K(z_{12})$ introduced already earlier. $K(z_{12})$ is expected to behave like a sufficiently narrow Gaussian on basis of the intuition about the behavior of coherent states.

Possible problems are also caused by the small values of y_{12} for which one might have $|G(1+iy_{12})| > 1$ implying the failure of the Schwartz inequality

$$|\langle \Psi_{z_1} | \Psi_{z_2} \rangle| \leq |\Psi_{z_1}||\Psi_{z_2}| \tag{104}$$

characterizing positive definite metric. The direct calculation of $G(1+iy)$ at the limit $y \rightarrow 0$ by using $\zeta(1+iy) \simeq 1/iy$ however gives

$$G(1) = 1 \ . \tag{105}$$

By a straightforward calculation one can also verify that $z = 1$ is a local maximum of $|G(z)|$. Note that the Jacobians do not affect the required inequality at all in case of eigen states.

It is easy to see that arbitrary small values of y_{12} are unavoidable. The estimate of Riemann for the number of the zeros of ζ in the interval $Im[s] \in [0,T]$ along the line $Re[s] = 1/2$ reads as

$$N(T) \simeq \frac{T}{2\pi}\left[log(\frac{T}{2\pi}) - 1\right] \ , \tag{106}$$

and allows to estimate the average density dN_T/dy of the zeros and to deduce an upper limit for the minimum distance y_{12}^{min} between two zeros in the interval T:

$$\begin{aligned} \frac{dN_T}{dy} &\simeq \frac{1}{2\pi}\left[log(\frac{T}{2\pi}) - 1\right] \ , \\ y_{12}^{min} &\leq \frac{1}{\frac{dN_T}{dy}} = \frac{2\pi}{\left[log(\frac{T}{2\pi}) - 1\right]} \rightarrow 0 \ \text{for} \ T \rightarrow \infty \ . \end{aligned} \tag{107}$$

This implies that arbitrary small values of y_{12} are unavoidable.

4.8 Super-conformal symmetry

Before considering super-conformal symmetry it is good to summarize the basic results obtained hitherto.

a) Conformal invariance as a gauge symmetry is possible only in the space $\mathcal{V}$ spanned by the eigen states associated with the zeros of ζ.

b) The hermiticity of the metric in the space spanned by the eigen states associated with the zeros of ζ is possible only if the zeros are on the critical line.

c) The requirement that the algebra spanned by the generators $L_{2Re[s]-1}$, $L_{1-2Re[s]}$ act as a dynamical symmetry algebra generating new zeros of ζ, forces the zeros to be on the critical line: in this case the generators in question reduce to L_0 and the dynamical symmetry reduces to a gauge symmetry.

One can say that the relationship of the conformal invariance to Riemann hypothesis is understood. Although super-conformal invariance does not seem to bring in anything new in this respect, it is still interesting to look whether conformal symmetry could be generalized to super-conformal symmetry. Certainly the basic idea about the action as gauge symmetry remains the same as well as the manner how subalgebra of conformal algebra acts as a dynamical symmetry algebra.

In the following various approaches to the problem of finding a super-conformal generalization of the dynamical system associated with the Riemann Zeta are discussed.

4.8.1 Simplest variant of the super-conformal symmetry

One can indeed identify a conformal algebra naturally associated with the proposed dynamical system. Note first that the generators of the ordinary conformal algebra

$$L_z = \Psi_z D^+ \tag{108}$$

generate conformal algebra with commutation relations ($[A,B] \equiv AB - BA$)

$$[L_{z_1}, L_{z_2}] = (z_2 - z_1) L_{z_1+z_2} \ . \tag{109}$$

Fermionic generators G_z satisfy the following anti-commutation and commutation relations:

$$\{G_{z_1}, G_{z_2}\} = L_{z_1+z_2} \ , \quad [L_{z_1}, G_{z_2}] = z_2 G_{z_1+z_2} \ , \ . \tag{110}$$

An explicit representation for the generators of the algebra extended to a super-algebra is obtained by introducing besides the bosonic coordinate t an anti-commuting coordinate θ. This means that the ordinary complex function algebra is replaced by the function algebra consisting of functions $f(t) + \theta g(t)$.

It is easy to verify that the generators defined as

$$L_z = t^z (D^+ + z\theta d_\theta) \ , \quad G_z = \frac{1}{\sqrt{2}} t^z (d_\theta + \theta D^+) \ . \tag{111}$$

satisfy the defining commutation and anti-commutation relations of the super conformal algebra. Notice that the definition of the operator $D^+ = L_0$ is not affected at all by the generalization and the eigenfunctions of D^+ come as doubly degenerate pairs consisting of a bosonic state Ψ_z and its fermionic partner $\Psi_z\theta$. Vacuum state however corresponds to the bosonic state since L_z and G_z do not annihilate the fermionic partner of the vacuum state.

The representation of this algebra as a gauge algebra is achieved in exactly the same manner as in the case of the ordinary conformal algebra. The gauge conditions for L_z are satisfied only by the bosonic eigen states so that actually nothing new seems to emerge from this generalization. The counterpart of the algebra generated by $L_{1-2Re[s]}$, $L_{2Re[s]-1}$ and L_0 is obtained by adding the generator G_0. Since any L_z commutes with G_0 the algebra closes. The requirement that this algebra acts as a symmetry in $\mathcal{V}$ implies Riemann hypothesis since the algebra reduces to that generated by L_0 and G_0 on the critical line. The super-symmetric variant of the theory is clearly somewhat disappointing exercise since it does not seem to bring anything genuinely new: even the space of the conformally invariant states remains the same.

4.8.2 Second quantized version of super-conformal symmetry

The following much more complex construction is essentially a construction of a second-quantized super-conformal quantum field theory for the super-symmetric system associated with D^+. It must be emphasized that this construction contains un-necessary complexities. In particular, the introduction of Kac Moody symmetry can be criticized since Kac Moody generators cannot annihilate physical states in the representation of the super-conformal symmetries as gauge symmetries in the space $\mathcal{V}$. It is however perhaps wise to keep also this option since it turn out to be of some value.

The extension of this algebra to super-conformal algebra requires the introduction of the fermionic generators G_z and $G^{\dagger}_{z}$. To avoid confusions it must be emphasized that following convention concerning Hermitian conjugation is adopted to make notation more fluent:

$$(O_w)^{\dagger} = O^{\dagger}_{\overline{w}} \ . \tag{112}$$

Fermionic generators G_z and $G^{\dagger}_{z}$ satisfy the following anti-commutation and commutation relations:

$$\{G_{z_1}, G^{\dagger}_{z_2}\} = L_{z_1+z_2} \ , \quad [L_{z_1}, G_{z_2}] = z_2 G_{z_1+z_2} \ , \quad \left[L_{z_1}, G^{\dagger}_{z_2}\right] = -z_2 G^{\dagger}_{z_1+z_2} \ . \tag{113}$$

This definition differs from that used in the standard approach [26] in that generators G_z and $G^{\dagger}_{z}$ are introduced separately. Usually one introduces only the the generators G_n and assumes Hermiticity condition $G_{-n} = G^{\dagger}_{n}$. The anti-commutation relations of G_z contain usually also central extension term. Now this term is not present as will be found.

Conformal algebras are accompanied by Kac Moody algebra which results as a central extension of the algebra of the local gauge transformations for some Lie group on circle or line [26]. In the standard approach Kac Moody generators are Hermitian in the sense that one has $T_{-n} = T^{\dagger}_{n}$ [26]. Now this condition is dropped and one introduces also the generators $T^{\dagger}_{z}$. In present case the counterparts for the generators $T^{\dagger}_{z}$ of the local gauge transformations act as translations $z_1 \rightarrow z_1 + z$ in the index space labelling eigenfunctions and geometrically correspond to the multiplication of Ψ_{z_1} with the function t^z

$$T^{\dagger}_{z_1}\Psi_{z_2} = t^{z_1}\Psi_{z_2} = \Psi_{z_1+z_2} \ . \tag{114}$$

These transformations correspond to the isometries of the Hermitian form defined by $G(z_{12})$ and are therefore natural symmetries at the level of the entire space of the eigenfunctions.

The commutation relations with the conformal generators follow from this definition and are given by

$$[L_{z_1}, T_{z_2}] = z_2 T_{z_1+z_2} \ \ , \quad \left[L_{z_1}, T^{\dagger}_{z_2}\right] = -z_2 T^{\dagger}_{z_1+z_2} \ \ , \tag{115}$$

The central extension making this commutative algebra to Kac-Moody algebra is proportional to the Hermitian metric

$$[T_{z_1}, T_{z_2}] = 0 \ \ , \quad \left[T^{\dagger}_{z_1}, T^{\dagger}_{z_2}\right] = 0 \ \ , \quad \left[T^{\dagger}_{z_1}, T_{z_2}\right] = (z_1 - z_2) G(z_1 + z_2) \ \ . \tag{116}$$

One could also consider the central extension $\left[T^{\dagger}_{z_1}, T_{z_2}\right] = G(z_1 + z_2)$, which is however not the standard Kac-Moody central extension.

One can extend Kac Moody algebra to a super Kac Moody algebra by adding the fermionic generators Q_z and $Q^{\dagger}_z$ obeying the anti-commutation relations ($\{A, B\} \equiv AB + BA$)

$$\{Q_{z_1}, Q_{z_2}\} = 0 \ \ , \quad \{Q^{\dagger}_{z_1}, Q^{\dagger}_{z_2}\} = 0 \ \ , \{Q_{z_1}, Q^{\dagger}_{z_2}\} = G(z_1 + z_2) \ \ . \tag{117}$$

Note that also Q_0 has a Hermitian conjugate $Q^{\dagger}_0$, and one has

$$\{Q_0, Q^{\dagger}_0\} \ \ = \ \ G(0) = -\frac{1}{2} \tag{118}$$

implying that also the fermionic counterpart of Ψ_0 has negative norm. One can identify the fermionic generators as the gamma matrices of the infinite-dimensional Hermitian space spanned by the eigenfunctions Ψ_z. By their very definition, the complexified gamma matrices $\Gamma_{\overline{z}_1}$ and Γ_{z_2} anti-commute to the Hermitian metric $\langle \Psi_{z_1} | \Psi_{z_2} \rangle = G(\overline{z}_1 + z_2)$.

The commutation relations of the conformal and Kac Moody generators with the fermionic generators are given by

$$\begin{array}{ll} [L_{z_1}, Q_{z_2}] = z_2 Q_{z_1+z_2} \ \ , & \left[L_{z_1}, Q^{\dagger}_{z_2}\right] = -z_2 Q^{\dagger}_{z_1+z_2} \ \ , \\ \left[T_{z_1}, Q^{\dagger}_{z_2}\right] = 0 \ \ , & [T_{z_1}, Q_{z_2}] = 0 \ \ . \end{array} \tag{119}$$

The non-vanishing commutation relations of T_z with G_z and non-vanishing anticomutation relations of Q_z with G_z are given by

$$
\begin{aligned}
\left[G_{z_1}, T^\dagger_{z_2}\right] = Q_{z_1+z_2} \ \ , \quad \left[G^\dagger_{z_1}, T_{z_2}\right] = -Q^\dagger_{z_1+z_2} \ \ , \\
\{G_{z_1}, Q^\dagger_{z_2}\} = T_{z_1+z_2} \ \ , \quad \{G^\dagger_{z_1}, Q_{z_2}\} = T^\dagger_{z_1+z_2} \ \ .
\end{aligned}
\tag{120}
$$

Super-conformal generators clearly transform bosonic and fermionic Super Kac-Moody generators to each other.

The final step is to construct an explicit representation for the generators G_z and L_z in terms of the Super Kac Moody algebra generators as a generalization of the Sugawara representation [26]. To achieve this, one must introduce the inverse $G^{-1}(z_a z_b)$ of the metric tensor $G(z_a z_b) \equiv \langle \Psi_{z_a} | \Psi_{z_b} \rangle$, which geometrically corresponds to the contravariant form of the Hermitian metric defined by G. Adopting these notations, one can write the generalization for the Sugawara representation of the super-conformal generators as

$$
\begin{aligned}
G_z &= \sum_{z_a} T_{z+z_a} G^{z_a z_b} Q^\dagger_{z_b} \ \ , \\
G^\dagger_z &= \sum_{z_a} T^\dagger_{z+z_a} G^{z_a z_b} Q_{z_b} \ \ .
\end{aligned}
\tag{121}
$$

One can easily verify that the commutation and anti-commutation relations with the super Kac-Moody generators are indeed correct. The generators L_z are obtained as the anti-commutators of the generators G_z and $G^\dagger_z$. Due to the introduction of the generators T_z, $T^\dagger_z$ and G_z, $G^\dagger_z$, the anti-commutators $\{G_{z_1}, G^\dagger_{z_2}\}$ do not contain any central extension terms. The expressions for the anti-commutators however contains terms of form $T^\dagger T Q^\dagger Q$ whereas the generators in the usual Sugawara representation contain only bilinears of type $T^\dagger T$ and $Q^\dagger Q$. The inspiration for introducing the generators T_z,G_z and $T^\dagger_z$, $G^\dagger_z$ separately comes from the construction of the physical states as generalized super-conformal representations in quantum TGD [F2]. The proposed algebra differs from the standard super-conformal algebra [26] also in that the indices z are now complex numbers rather than half-integers or integers as in the case of the ordinary super-conformal algebras [26]. It must be emphasized that one could also consider the commutation relations $\left[T^\dagger_{z_1}, T_{z_2}\right] = iG(z_1 + z_2)$ and they might be more the physical choice since $z_2 - z_1$ is now a complex number unlike for ordinary super-conformal representations. It is not however clear how and whether one could construct the counterpart of the Sugawara representation in this case.

Imitating the standard procedure used in the construction of the representations of the super-conformal algebras [26], one can assume that the vacuum state is annihilated by *all* generators L_z irrespective of the value of z:

$$L_z|0\rangle = 0 \ , \quad G_z|0\rangle = 0 \ . \tag{122}$$

That all generators L_z annihilate the vacuum state follows from the representation $L_z = \Psi_z D_+$ because D_+ annihilates Ψ_0. If G_0 annihilates vacuum then also $G_z \propto [L_z, G_0]$ does the same.

The action of $T_z^\dagger$ on an eigenfunction is simply a multiplication by t^z: therefore one cannot require that T_z annihilates the vacuum state as is usually done [26]. The action of T_0 is multiplication by $t^0 = 1$ so that T^0 and $T_0^\dagger$ act as unit operators in the space of the physical states. In particular,

$$T_0|0\rangle = T_0^\dagger|0\rangle = |0\rangle \ . \tag{123}$$

This implies the condition

$$\left[T_0, T_z^\dagger\right] = izG(z) = 0 \tag{124}$$

in the space of the physical states so that physical states must correspond to the zeros of ζ and possibly to $z = 0$. Thus one can generate the physical states from vacuum by acting using operators $Q_z^\dagger$ and $T_z^\dagger$ with $\zeta(z) = 0$. If one requires that the physical states also have real and positive norm squared, only the zeros of ζ on the line $Re[s] = 1/2$ are allowed. Hence the requirement that a unitary representation of the super-conformal algebra is in question, forces Riemann hypothesis.

It is important to notice that $T_z^\dagger$ and $Q\dagger_z$ cannot annihilate the vacuum: this would lead to the condition $G(z_1 + z_2) = 0$ implying the vanishing of $\zeta(z_1 + z_2)$ for any pair $z_1 + z_2$. One can however assume that Q_z annihilates the vacuum state

$$Q_z|0\rangle = 0 \ . \tag{125}$$

The realization of these conditions in case of super-conformal algebra is achieved by mapping the eigen states Ψ_s to $\Psi_{\zeta(s)}$, acting to these states by the generators of the algebra and mapping the resulting state (which vanishes for zeros of ζ) back to a state proportional to $\Psi_{\zeta^{-1}(\zeta(s)+z)}$. It must be however emphasized that for Kac Moody generators not annihilating the vacuum state the action is not well-defined.

This inspires the hypothesis that only the generators with conformal weights $z = 1/2 + iy$ generate physical states from vacuum realizable in the space of the eigenfunctions Ψ_z and their fermionic counterparts. This means that the action of the bosonic generators $T^{\dagger}_{1/2+iy}$ and fermionic generators $Q^{\dagger}_0$ and $Q^{\dagger}_{1/2+iy}$, as well as the action of the corresponding super-conformal generators $G^{\dagger}_{1/2+iy}$, generates bosonic and fermionic states with conformal weight $z = 1/2 + iy$ from the vacuum state:

$$|1/2 + iy\rangle_B \equiv T^{\dagger}_{1/2+iy}|0\rangle \ , \quad |1/2 + iy\rangle_F \equiv Q^{\dagger}_{1/2+iy}|0\rangle \ . \tag{126}$$

One can identify the states generated by the Kac Moody generators $T^{\dagger}_z$ from the vacuum as the eigenfunctions Ψ_z. The system as a whole represents a second quantized super-symmetric version of the bosonic system defined by the eigenvalue equation for D^+ obtained by assigning to each eigenfunction a fermionic counterpart and performing second quantization as a free quantum field theory.

It should be noticed that the ordinary Super Kac-Moody and super-conformal algebras with generators O_n labelled by integers $n > 0$ generate zero norm states from any state $|z\rangle$ with $Re[z] = 0$ or $Re[z] = 1/2$ ($G(n_1 + n_2) = 0$). Thus ordinary super-conformal invariance holds true as gauge invariance. It is possible (although perhaps not absolutely necessary) to restrict the real parts of the conformal weights of the generators to be non-negative.

4.8.3 Is the proof of the Riemann hypothesis by reductio ad absurdum possible using second quantized super-conformal invariance?

Riemann hypothesis is proven if all eigenfunctions for which the Riemann Zeta function vanishes, correspond to the states having a real and positive norm squared. The expectation is that super-conformal invariance realized in some sense excludes all zeros of ζ except those on the line $Re[s] = 1/2$. The problem is to define precisely what one means with super-conformal invariance and one can generate large number of reduction ad absurdum type proofs depending on how super-conformal invariance is assumed to be realized. The following considerations are completely independent of the already described and more recent realization of the super-conformal gauge invariance by applying ζ and its inverse to the conformal weights of the eigen states. I have kept this material because I feel that it might be unwise to to throw it way yet.

The most conservative option is that super-conformal invariance is realized in the standard sense. The action of the ordinary super-conformal generators L_n, and G_n, $n \neq 0$ on the vacuum states $|0\rangle_{B/F}$ or on any state $|1/2+iy\rangle_{B/F}$ indeed creates zero norm states as is obvious from the vanishing of the factor $sin(\pi z_{12}) = sin(\pi(x_1+x_2))$ associated with the inner inner products of these states. Thus the zeros of ζ define an infinite family of ground states for the representations of the ordinary super-conformal algebra. A generalization of this hypothesis is that the action of L_n and G_n, $n \neq 0$, on any state $|w\rangle_{B/F}$, $\zeta(w) = 0$, creates states which are orthogonal zero norm states. This implies $\zeta(n+2Re[w]) = 0$ for all values of $n \neq 0$ and, since the real axis contains zeros of ζ only at the points $Re[s] = -2n$, $n > 0$, leads to a reductio ad absurdum unless one has $Re[w] = 1/2$. Thus the proof of the Riemann hypothesis would reduce to showing that the action of the ordinary super-conformal algebra generates mutually orthogonal zero norm states from any state $|w\rangle_{B/F}$ with $\zeta(w) = 0$. The proof of this physically plausible hypothesis is not obvious.

One can imagine also other strategies. The minimal requirement is certainly that some subalgebra of the super-conformal algebra generates a space of states satisfying the Hermiticity condition. The quantity

$$\Delta(\overline{w}_1 + w_2) \equiv \langle w_1|w_2\rangle - \overline{\langle w_2|w_1\rangle} = G(\overline{w}_1 + w_2) - \overline{G(\overline{w}_2 + w_1)} \quad (127)$$

must define the conformal invariant in question since this quantity must vanish in the space of the physical states for which the metric is Hermitian. This requirement does not however imply anything nontrivial for the ordinary conformal algebra having generators L_n and G_n: for $Re[w] \neq 1/2$ the condition is indeed satisfied because $G(n+2Re[w])$ does *not* satisfy the Hermiticity condition for any value of n.

One can try to abstract some property of the states associated with the zeros of ζ on the line $Re[s] = 1/2$. The generators $L_{1/2-iy}$ and $G_{1/2-iy}$ generate zero norm states from the states $|1/2+iy\rangle_{B/F}$, when $1/2+iy$ corresponds to the zero of ζ on the line $Re[s] = 1/2$. One can try to generalize this observation so that it applies to an arbitrary state $|w\rangle_{B/F}$, $\zeta(w) = 0$. The generators $L_{1-\overline{w}}$ and $G_{1-\overline{w}}$ certainly generate zero norm states from the states $|w\rangle_{B/F}$. Also the Hermiticity condition holds true identically and does not have nontrivial implications. One can however consider alternative generalizations by assuming that

a) either the generators $L_{\overline{w}}$ and $G_{\overline{w}}$ or

b) $L_{1/2+iy}$ and $G_{1/2+iy}$ generate from the states $|w\rangle_{B/F}$, $\zeta(w)=0$ states satisfying the Hermiticity condition.

These two hypothesis lead to two versions of a reductio ad absurdum argument. Suppose that w is a zero of ζ. This means that the inner product of the states $Q_0^\dagger|0\rangle$ and $Q_w^\dagger|0\rangle$ and thus also $\Delta(w)$ vanishes:

$$\langle 0|Q_0 Q_w^\dagger|0\rangle = 0 \ , \quad \Delta(w) = 0 \ . \tag{128}$$

a) By acting on this matrix element by the conformal algebra generator $L_{\overline{w}}$ (which acts like derivative operator on the arguments of the should-be Hermitian form), and using the fact that $L_{\overline{w}}$ annihilates the vacuum state, one obtains

$$\langle 0|Q_0 Q_{\overline{w}+w}^\dagger|0\rangle = G(w+\overline{w}) \ . \tag{129}$$

The requirement $\Delta(w+\overline{w})=0$ implies the reality of $G(w+\overline{w})$ and thus the condition $Re[w]=1/2$ leading to the Riemann hypothesis. Note that the argument implying the reality of $G(w+\overline{w})$ assumes only that L_w annihilates vacuum.

If this line of approach is correct, the basic challenge would be to show on the basis of the super-conformal invariance alone that the condition $\zeta(w)=0$ implies that the generators $L_{\overline{w}}$ and $G_{\overline{w}}$ generate new ground states satisfying the Hermiticity condition.

b) An alternative line of argument uses only the invariance under the generators $L_{1/2+iy}$ associated with the zeros of ζ, and thus certainly belonging to the conformal algebra associated with the physical states. By applying the generators $L_{1/2+iy_i}$ to the the matrix element $\langle 0|Q_0 Q_w^\dagger|0\rangle = 0$ and requiring that Hermiticity is respected, one can deduce that $G(w+1/2+iy_i)$ satisfies the Hermiticity condition. Hence the line $Re[s]=Re[w]+1/2$, and by the reflection symmetry also the line $Re[s]=1/2-Re[w]$, contain an infinite number of zeros of ζ if one has $Re[w]\neq 1/2$. By repeating this process once for the zeros on the line $Re[s]=1/2-Re[w]$, one finds that the lines $Re[s]=1-Re[w]$ and $Re[s]=Re[w]$ contain infinite number of the zeros of ζ of form $w_{ij}=w+i(y_i+y_j)$, where y_i and y_j are associated with the zeros of ζ on the line $Re[s] = 1/2$. By applying this two-step procedure repeatedly, one can fill the lines $Re[s] = Re[w]$,$1 - Re[w]$, $1/2 - Re[w]$, $1/2 + Re[w]$ with the zeros of ζ.

4.9 p-Adic version of the modified Hilbert-Polya hypothesis

Rather interestingly, the dynamical model generalizes in straightforward manner to the p-adic context. The first problem encountered in p-adicization of the results obtained thusfar relates to the definition of the p-adic eigenvalue problem. The functions t^{x+iy} do not exist p-adically unless one assumes that t is integer valued, p_1^{iy} defines Pythagorean phase and p_1^x exists for every prime. For arbitrary rational value of $x = m/n$ this requires that $p_1^{m/n}$ exists for every p_1 in the algebraic extension associated with R_p. These conditions also guarantee the existence of the p-adic Riemann Zeta.

The basic requirement is that orthogonality conditions lead to the vanishing of p-adic Riemann Zeta. This is achieved if one defines p-adic inner product simply as the sum

$$\langle \Psi_{z_1} | \Psi_{z_2} \rangle = \sum_n n^{z_{12}} = \zeta(z_{12}) \ , \quad z_{12} = x_1 + x_2 + i(y_2 - y_1) \ . \tag{130}$$

It is important to notice that p-adic Riemann Zeta is formally the inverse of the real Riemann Zeta: this is implied by the requirement that p-adic Riemann Zeta vanishes for $z = -2n$ and also suggested by adelic formula.

This definition means that in p-adic case the differential operator D is simply the formal differential operator $L_0 = td/dt$, that is free scaling operator without any interaction term and thus having as its eigenvalues exponents $x = x + iy$. $D^\dagger = -D$ obviously holds true. A possible interpretation is that conformal invariance is broken in real case by the emergence of the interaction potential $V(t)$ whereas in p-adic case this symmetry is unbroken. The study of p-adic Riemann Zeta indeed leads to the general view that infinite hierarchy of breakings of conformal symmetry occurs as p increases and destroys zeros of Riemann zeta so that at the limit $p \to \infty$ leaves only the zeros of Riemann Zeta located at line $x = 1/2$ remain.

What is fascinating is that for the representations of Super Virasoro only half-odd and integer eigenvalues of L_0 are possible in case that eigenvalues are real. Indeed, for Neveu-Schwartz type representations fermionic supersymmetry generators are labelled by half-odd integers. In quantum TGD these representations combine to form a larger algebra in which both conformal and super-conformal generators are labelled by half-integer valued conformal weight [B2, B3]. This would mean that $x = n/2$ are the only possible values of x and this would imply Riemann hypothesis since $x = 0$ and $x = 1$ are included by the previous considerations. The reason for half-odd integers is basically that the representations functions $z^{n/2}$ define representations of double-fold covering of Lorentz group acting as Möbius transformations of complex plane. This suggests that spin-statistics theorem allowing only single and double valued representation function is involved with Riemann hypothesis.

In p-adic case the requirement that probability density and thus also p-adic norm are *ordinary* p-adic numbers implies $x = n/2$. This does not however prove Riemann hypothesis unless all Ψ_z orthogonal to Ψ_0 belong to the state space. For a general rational value $x = m/n$ of x the values of the p-adic Riemann Zeta are in the algebraic extension and the number of vanishing conditions is much larger than the number of coordinate variables (x and y) so that with the rigour used by physicist one can conclude that the conditions are very probably not satisfied. If one could prove that irrational values of x do not belong to the spectrum of the operator D, one would be quite near to the proof of Riemann hypothesis if Local-Global principle is assumed. Super-conformal invariance might be the key for proving that only the values $x = n/2$ are possible.

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